KELLY-ULAM CONJECTURE AND GRAPH NUMBERING

The Creator invented the undirected graphs.
Everything else is the artificial product of mankind.[1]

K-U property is a graph property given in the heaven.[2]

Dedicated to
Doctor Takeo Ogata
The first real scientist in the country village,
who had opened a small dispensary there in the mid 1950',
and devoted his life to the medical care of villagers.

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Abstract

Kelly-Ulam Conjecture has been a long outstanding problem for about 60 years. It was posed by P.J. Kelly in 1942 and recited later by S.M. Ulam. A vertex-deleted subgraph G-v is the subgraph of G induced by $V(G)-\{v\}$. The conjecture states that two graphs G and H with at least three vertices are isomorphic iff they have a same set of vertex-deleted subgraphs. This problem has been referred to as graph reconstruction problem due to the title of Harary's paper, "On the reconstruction of a graph from a collection of subgraphs", which pointed out an alternative point of view such that a graph G is reconstructible from the collection of vertex-deleted subgraphs. The problem had been actively studied from 1960 through early 1990'. According to the preface written by Erdös for a book published to celebrate the 60-th anniversary of Tutte, a respectful mathematician said "There are three diseases in graph theorists. The first is four-color-disease, the second is reconstruction-disease, and the third hamiltonian-disease." Despite of the great efforts of so many researchers, only a few subclasses of graphs are known to be reconstructible, such as trees, regular graphs, and disconnected graphs. McKay verified practically that graphs with up to 11 vertices are determined uniquely by their subgraphs. We introduce a graph numbering called Ψ numbering system, which is a kind of Gödel numbers, and try to solve the conjecture positively in general. We come up with some new complete graph invariants expressed as Ψ numbers. However our attempt eventually fails and we conclude that it is possible that a K-U counterexample exists. Instead we get a simple proof regarding the edge-version K-U conjecture.

1 "God created the natural numbers. Everything else is the work of man." - Kronecker.

2 To reconstruct a graph from its vertex-deleted subgraphs is somewhat resemble to recollect the very vague memory of our lost paradise from where we had been expelled.

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1. Introduction

We consider that the world is a place where additivity generally rules, in other words we think that the world where we live is a physical world, and additivity is the first property of material. This comes from a simple fact that two particles cannot occupy a same spatial position at a same time.[4] Due to this we can decompose things and recompose them. Or we can classify things by applying abstraction and reduction in dual directions. Further we know that this operation is applicable not only to things but also to processes. Division of labors was the beginning of the civilization. The statement "The whole is a sum of parts" represents this property. Apparently this proposition is trivially true for Set Theory. However it is not necessarily true in Graph Theory. The simplest counterexample pair is $2K_1$ and K_2.[5] Those graphs have the same collection of vertex-deleted subgraphs $\{K_1,K_1\}$, but not isomorphic.

Paul Joseph Kelly conjectured in 1942 that with respect to two undirected graphs G and H of order ≥ 3, if for each vertex i, a pair of vertex-deleted subgraphs $G - i$ and $H - i$ are isomorphic, then G and H are isomorphic [18]. Stanislaw M. Ulam recited the conjecture in 1960 [39]. We may call it K-U Conjecture or simply K-U. The conjecture is equivalent to the proposition, "Graphs of order ≥ 3 are reconstructible from the collection of their vertex-deleted subgraphs", therefore it is usually referred to as Reconstruction Conjecture [14].

It is easy to see that regular graphs are reconstructible. Take an arbitrary vertex-deleted subgraph, add a vertex to it and supplement lacking edges. Kelly proved in 1957 that trees are reconstructible from their subgraphs [19]. Erdös and Rényi showed the fact that almost all graphs are reconstructible in 1963 [9]. Manvel proved in 1976 that disconnected graphs are reconstructible [28]. A planar graph is maximal if no edge can be added without losing its planarity. Fiorini and Lauri proved maximal planar graphs are reconstructible in 1981 [10]. As well separable graphs with no endvertices are reconstructible, where a separable graph is a connected graph to be disconnected by removing a vertex of it, and an endvertex is a vertex of degree 1 [3].

Frank Harary posed the edge version of K-U Conjecture in 1964, stating that a graph G with at least four edges is reconstructible from its edge-deleted subgraphs, where edge-deleted subgraph $G - e$ is the subgraph of G obtained by removing an edge e from G [14]. It can be said that the edge-reconstruction is easier than the vertex-reconstruction. In fact Greenwell showed in 1971 that if G has no isolated vertices then the vertex-deleted subgraphs are reconstructible from its edge-deleted subgraphs [12].[6] As the restriction of the theorem is quite trivial, we can say that if a graph G is vertex-reconstructible then G is also edge-reconstructible. Several graph classes were added to the list of the edge-reconstructible graphs, including planar graphs with minimum degree 5 [22], 4-connected planar graphs [11], claw-free graphs [8] and hamiltonian graphs of sufficiently large order [31].

Those results are detailed in Bondy and Hemminger [5], Nash-Williams [30], Ellingham [7], Bondy [4], and so many other survey articles and text books [1]. After all of those, Lauri wrote in 1992 like the following [24].

"...the list of classes of reconstructible or edge-reconstructible graphs falls far short of exhausting all possibilities. If only regarded as step-by-step efforts at obtaining an ultimate proof of the reconstructiblility of all graphs, then the outlook is bleak — the cases solved are few and the techniques used to tackle one class of graphs do not generalize to other classes. It seems hardly likely that by working laboriously in this fashion at successive classes of graphs one can ultimately prove the Reconstruction Conjecture for all graphs. So why do graph theorists persist in nibbling away at this mighty and unyielding problem?".

An basic difficulty of K-U Conjecture is the existence of pseudosimilar vertices. An automorphism of graph G is an isomorphism of G itself and two vertices u and v are similar if for some automorphism Φ of G, $\Phi(u) = v$. If u and v are similar, then clearly $G - u$ and $G - v$ are isomorphic. However the converse is not necessarily true. Harary and Palmer pointed out this crucial fact in 1966 [16]. If $G - u$ and $G - v$ are isomorphic but u and v are not similar, then they are said to be pseudosimilar. Obviously a graph G cannot have all its vertices mutually pseudosimilar, because otherwise G

[4] "My mother said, 'Even you, Paul, can be in only one place at one time.'". - P. Erdös.

[5] K_n is a complete graph of order n and kG is a graph such as a union of k graphs G. Incidentally $K_{m,n}$ is a complete bipartite graph with m and n vertices parts.

[6] The converse, i.e., the edge-deleted subgraphs are reconstructible from its vertex-deleted subgraphs is still open. It turns to the most urgent subject regarding K-U conjecture now.

would be regular and a regular graph cannot have pseudosimilar vertices. Therefore to find the largest set of mutually pseudosimilar vertices becomes a focal point of the problem [20][23]. Stockmeyer showed in 1977 a construction of non-reconstructible directed graphs involving tournaments such that every vertex has a pseudosimilar mate [34]. However it is quite unclear how pseudosimilarity relates to K-U problem in undirected graphs. Anyway pseudo-similarity forms a dense and wide bush zone of the problem [24].

So far so many graph invariants are proved to be reconstructible from a collection of vertex-deleted subgraphs, including number of edges, degree sequence, maximum size of a matching, number of isolated vertices, number of 1-factors, number of spanning trees, chromatic number, rank polynomial, chromatic polynomial, flow polynomial, determinant of an adjacent matrix, characteristic polynomial, number of hamiltonian paths and cycles, idiosyncratic polynomial, and so on [6][25][33][35]. Since graph isomorphism is intersecting with Group Theory, K-U problem has some relevance to Algebra. W. T. Tutte and Kocay contributed here remarkably [36][37][21].

As was described above, a very few subclasses of graphs are known to be reconstructible. However once it is established, then it turns out generally that information of the full collection of subgraphs is rather redundant than that is required to determine the graph. In 1985 Harary and Plantholt introduced a concept of reconstruction number such as the smallest number of vertex-deleted subgraphs of a graph which is sufficient to reconstruct the graph [17]. Myrvold proved that almost every graph has reconstruction number 3 in 1988 [29], and it was certified by Bollobás in 1990 independently [2]. The isomorph-reduced deck of G is a set containing a single member of each isomorphism type of vertex-deleted subgraphs of G. A strong form of K-U Conjecture is that a graph of order ≥ 4 is uniquely determined by its isomorph-reduced deck [14]. B. D. McKay verified practically that K-U Conjecture is valid even in the strong form for all graphs of order ≤ 11 and some larger special graphs such as triangle-free graphs, square-free graphs, bipartite graphs, and so on [27]. The total CPU time spent was about one year with SUN workstations.

Now we understand that K-U Conjecture is truly difficult to solve. It looks as if there is no way to prove it. On the other hand, validity of the conjecture is intuitively almost sure, since it is very unlikely that two graphs comes to be non-isomorphic even though every subgraphs of them exactly coincide. How can we break through this stalemate? The first step may be to try reductio ad absurdum. Let the negation of K-U Conjecture be X. If X is true, then of course K-U is invalid. Figuratively speaking it is probable that we cannot find any contradiction in X, while there is no two worlds together, one is K-U-valid and another K-U-invalid. Going on this line, we may eventually find any partial conditions which would make some subclasses of graphs reconstructible such as regular graphs, disconnected graphs and so on. However it seems that there is no straight path on this course to reach to a simple solution but an innumerably ramified maze with blind alleys to every directions.

Mathematical induction must be tried next. Assume K-U is true for all graphs of order $\leq k$. If we can prove K-U for graphs of order $k + 1$, then we are done, since we know that K-U is true for small graphs. However we found that this approach is almost of no use for K-U proofs, though the invalidity of K-U for directed graphs comes visible in this way. The problem here is that there is no common labelings being a ladder to the upper floor in the vertex-deleted subgraphs. On the other hand, if those subgraphs of G and H are isomorphic over some total labelings L of G and L' of H, then it turns out that G and H are trivially isomorphic. Undoubtedly we have to give some common labeling to the vertex-deleted subgraphs. How can we perform this? Of course it is almost equivalent to solve the reconstruction problem itself. Accordingly we have to say that it is impossible or very hard to compose a common labeling L for vertex-deleted subgraphs.

What we found at last is a graph numbering. We call this graph numbering the Ψ numbering system which is simply a kind of Gödel numbering or a variation of Gödel numbers. As far as we know there was no such an attempt to adopt Gödel number to Graph Theory, though Gödel numbers were broadly employed in Complexity Theory to solve isomorphism problems of programming systems. [32][13][38]. We will provide three kinds of Ψ numbers.[7] The first is called labeled Ψ numbers $\Psi(L)$, and the second unlabeled Ψ numbers $\Psi_U(G)$. The third is a blend of the two numbers and called natural Ψ numbers $\Psi_N(G)$. We raise a theorem stating that given two graphs G and H, $\Psi_N(G)$ is equal to $\Psi_N(H)$ if and only if $\Psi_U(G)$ equals $\Psi_U(H)$. We try to prove it by checking all possible labelings of two graphs.

[7] In Section 5 We introduce the fourth variation of Ψ numbers called supernatural Ψ numbers. Finally in Section 6 we provide four additional edge-version Ψ numbers called edge-connected Ψ numbers, edge-labeled Ψ numbers, edge-deck Ψ numbers, and edge-fragment-deck Ψ numbers.

2. Υ Numbering System

All through this paper, we mostly follow the terminology given by Harary [15]. A graph $G(V,E)$ consists of a finite non-empty set $V = V(G)$ of n vertices with prescribed set $E = E(G)$ of m unordered pairs (u,v) of distinct vertices u and $v \in V$. Each pair $x = (u,v)$ of vertices in E is called an edge of G and written uv. Two graphs G and H are isomorphic iff there exists a mapping Φ: $V(G) \to V(H)$ such that $uv \in E(G)$ iff $\Phi(u)\Phi(v) \in E(H)$, and written $G \cong H$. An invariant of graph G is a number associated with G which has the same value for any graph isomorphic to G. An invariant or a set of invariants is complete if it determines a graph up to isomorphism. A vertex-deleted subgraph $G - v$ is the subgraph of G induced by $V - \{v\}$. As well k-vertex-deleted subgraph $G - S$ is the subgraph of G induced by $V - S$, where a subset of vertices $S \subset V$ and $|S| = k$. We fix n at the order of graphs as well as m at the size of the graphs.

A sequence is an ordered set of elements and σ_k denotes the natural number sequence $(1,2,...,k)$.[8] Let an arbitrary sequence of length k be S. The sequence S can be regarded as a mapping from $\sigma_k = (1,2,...,k)$ to S itself, hence $S(i)$ denotes the i-th element in the sequence S. A concatenation of two sequences a and b is denoted $a + b$. As well we use a subtraction $a - b$ of two sequences a and b, where $a - b$ is the ordered set of elements $x \in a$ and $x \notin$ b. Of course $a - b$ preserves the ordering in the sequence a. We may sometimes abuse the element e for a singleton sequence (e).

A graph G is labeled when n vertices are distinguished from one another by names such as $p_1, p_2, ..., p_n$. However we restrict the labeling of graphs from now on as follows. A labeling L of graph G is a mapping from a vertex set $V = \{v_1, v_2, ..., v_n\}$ onto a set σ of distinct natural numbers, where $L(v_i) = j \in \sigma$. We fix the set σ at the natural number sequence $\sigma_n = (1,2,...,n)$ unless it is mentioned explicitly. We call such a labeling rule the natural labeling scheme. G_L denotes the graph G labeled by L, and L^G denotes the set of all labelings L of G.

Let x be an arbitrary sequence of distinct natural numbers $i \in \sigma_n$, and the set of all such number sequences x of length $\leq n$ be Σ^n. Σ^n_k denotes a subset of Σ^n such as $\Sigma^n_k = \{x \mid x \in \Sigma^n, |x| \leq n - k\}$ and $k \leq n$. The complementary number sequence $\sigma_n - x$ of a number sequence x is denoted $\bar{x} = \sigma_n - x$, and $\bar{x}(i)$ denotes the i-th number in the number sequence $\bar{x}$. Suppose a number sequence $x \in \Sigma^n_2$ and the vertex subset $S \subset V$ such as $L(S) = x$,[9] then $G_L(x)$ denotes k-vertex-deleted subgraph $G_L - S$, where $k = |x|$. $P(i)$ denotes the i-th prime number.

$$\Psi(K_1) := 1,\ \Psi(2K_1) := 3,\ \Psi(K_2) := 5, \text{ and}$$

$$\Psi(G_L(x)) := \prod_{i=1}^{r} P(i)^{\Psi(G_L(x+\bar{x}(i)))}, \text{ where } r = |\bar{x}| \text{ and } x \in \Sigma^n_2.$$

$$\Psi(L) := \Psi(G_L(f)), \text{ where } L \in L^G \text{ and } G_L \text{ is a labeled graph of } G \text{ labeled by } L.$$

$$\Psi(G) := \Psi(L_0) \geq \Psi(L) : {}^{\forall}L \in L^G, \text{ where } G \text{ is an unlabeled graph.}$$

$\Psi(G_L(x))$ of k-vertex-deleted subgraphs $G_L(x)$ is defined recursively for all number sequences $x \in \Sigma^n_2$ and $\Psi(L)$ is the case when x is the empty set ϕ. We call a $\Psi(L)$ a labeled Ψ number of G. Apparently the computation of a $\Psi(L)$ forms a tree such that its nodes are Ψ numbers $\Psi(G_L(x))$ in some computation level $0 \leq i \leq n - 2$. We call such a computation tree a Ψ-tree. Every node in level i is a vertex-deleted subgraph of its parent node, and the number of edges (excluding the edge incident with the parent node) of an inner node is $n - i$. Every leaves are either $2K_1$ or K_2. We call the set of all formulas defined above and appeared in a Ψ-tree a Ψ-formulas. Note that there is only one Ψ-tree, and then only one Ψ-formulas for all graphs of some fixed order n, although each actual values of Ψ numbers are different. An unlabeled graph G of order n has $n!$ $\Psi(L)$ corresponding to each labeling L of G, and $\Psi(G)$ is the maximum value among them. The number sequence $x \in \Sigma^n_2$ represents the set of deleted vertices in the upper levels in that order, and works as an indicator on a Ψ-tree. Whenever a vertex v of label $l = L(v)$ is deleted, the number l is appended at the tail of the number sequence x and the length of x increases by one. Every number sequence $x \in \Sigma^n_2$ appears exactly once at a fixed position on a Ψ-tree as the node $\Psi(G_L(x))$. Note that x designates the deleted vertices and $\bar{x}$ represents the remained vertices in the k-vertex-deleted subgraph $G_L(x)$.

[8] Note that "ordered set" here is quite different from "partially ordered set". This is just a set of successive elements.

[9] We permit to write $\Phi(s) = s'$ for a mapping $\Phi : \alpha \to \beta$ even in the case where s and s' are the subsets $s \subseteq \alpha$ and $s' \subseteq \beta$, as well as the case where s and s' are the elements $s \in \alpha$ and $s' \in \beta$. In the case of $L(S) = x$, S is a subset and x is a sub-sequence, i.e., an ordered subset. Hence the equality is valid in our convention. If either/both of s or/and s' are sequences, then of course the order must be preserved/coincident.

3. Y **Number Theorem**

A mapping from a finite set onto itself is called a permutation. An automorphism Φ of a graph G is an isomorph mapping $V \to V$, and automorphisms of G forms the automorphism group $\Gamma(G)$. Suppose two labelings L_1 and L_2 of G. Since a labeling L of G is a mapping $V \to \sigma_n$, there is a permutation Φ on V such as

$$V \xrightarrow{L_1} s_n \xrightarrow{I_n} s_n \xrightarrow{L_2^{-1}} V,$$

where I_n is the identity on σ_n and $\Phi = L_2^{-1} I_n L_1 = L_2^{-1} L_1$. If the permutation Φ is an automorphism, then we say that L_1 and L_2 is a label-automorphism of G (over natural labeling scheme). Note that if $\Phi = L_2^{-1} L_1$ is an automorphism of G, then the permutation $\Phi^{-1} = L_1^{-1} L_2$ is also an automorphism. Therefore, if (L_1,L_2) is a label-automorphism, then so is (L_2,L_1). Further it is easily to be certified that if (L_1,L_2) and (L_2,L_3) are label-automorphisms, then (L_1,L_3) is a label-automorphism, too. Consequently the label-automorphism relation is an equivalence relation and it partitions the set L^G of all labelings L of G into equivalence classes. Note that all through this paper labelings are denoted like L_* and other mappings except identity I_n are written in upper case Greek.[10] Hereafter "$\to$" is assumed to be always bijective.

3.1. *Y(L) and Y(L') are equal iff labelings L and L' of graph G is a label-automorphism of G.*

Proof: The statement is trivially true for graphs of order ≤ 2. Then assume the order of $G \geq 3$. If L and L' is a label-automorphism, then by the definition, there is a permutation $\Phi = L'^{-1} I_n L$ on V such that Φ is an isomorph mapping $V \to V$. Let the Ψ-trees of L and L' be Ψ_1 and Ψ_2 respectively. Consider subgraphs $G_L(x)$ and $G_{L'}(x)$ which have a same number sequence x as their argument. Positions of these subgraphs on Ψ-trees are decided uniquely by x, and each x is coded by L or L' respectively. Consequently for all number sequences $x \in \Sigma_2^n$, the isomorph mapping Φ maps $G_L(x)$ in Ψ_1 to the corresponding subgraph $G_{L'}(x)$ in Ψ_2 through the identity mapping I_n. Hence for all $x \in \Sigma_n^2$, $\Psi(G_L(x)) = \Psi(G_{L'}(x))$, and $\Psi(L) = \Psi(L')$. The converse. Assume $\Psi(L) = \Psi(L')$. Then for all $x \in \Sigma_2^n$, $\Psi(G_L(x))$ and $\Psi(G_{L'}(x))$ must be exactly coincident by the definition of Ψ numbers. Consider a permutation Φ on V decided by L and L' such as $\Phi = L'^{-1} I_n L$. Let y be a number sequence in Σ_2^n such as $|y| = n - 2$, then for all $y \in \Sigma_2^n$, $\Psi(G_L(y)) = \Psi(G_{L'}(y)) = 3$ or 5, and $\Phi(L^{-1}(y)) = L'^{-1}(y)$. Take vertex pairs w_1 and w_2 corresponding to y in the graph G mapped by L and L' respectively, where $L(w_1) = L(w_2) = y$. Since $\Phi(w_1) = w_2$ and $\Psi(G_L(L(w_1))) = \Psi(G_{L'}(L'(w_2))) = 3$ or 5, it comes to be that the subgraph of G_L induced by x_1 is a $2K_1$ (K_2) iff the subgraph of $G_{L'}$ induced by w_2 is a $2K_1$ (K_2 respectively). Hence the permutation Φ is an isomorph mapping: $V \to V$, and (L,L') is a label-automorphism (over natural labeling scheme). ■

3.2 *Two graphs G and H are isomorphic iff they have a same labeled* Y *number such as Y(L) equals Y(L').*

Proof: The statement is trivially true for graphs of order ≤ 2. Then assume the order of graphs ≥ 3. Suppose that two graphs G and H are isomorphic. Then there is an isomorph mapping Φ: $V(G) \to V(H)$. Let a labeling of G be L. We show that there is a labeling L' of H corresponding to L such that $\Psi(L) = \Psi(L')$. Suppose $\Phi = \Phi L^{-1} L = L'^{-1} L = L'^{-1} I_n L$ and $L' = L\Phi^{-1}$, where L' is a labeling of H. Let the Ψ-formulas of L and L' be Ψ and Ψ' respectively. For all number sequences $x \in \Sigma_2^n$, Φ maps x in Ψ to the same x in Ψ' through the identity mapping I_n. Consequently every subgraph $G_L(x)$ in Ψ is mapped to the corresponding subgraph $H_{L'}(x)$ in Ψ' by the isomorph mapping Φ. Hence for all $x \in \Sigma_2^n$, $\Psi(G_L(x)) = \Psi(H_{L'}(x))$, and this yields $\Psi(L) = \Psi(L')$. Next assume there are labelings L of G and L' of H such as $\Psi(L) = \Psi(L')$. Then for all $x \in \Sigma_2^n$, $\Psi(G_L(x))$ and $\Psi(H_{L'}(x))$ must be exactly coincident by the definition of Ψ numbering system. Consider a mapping Φ: $V(G) \to V(H)$ decided by L and L' such that $\Phi = L'^{-1} I_n L$. Let y be a number sequence in Σ_2^n such as $|y| = n - 2$. Take vertex pairs w in the graph G and w' in H mapped by L and L' respectively corresponding to y, where $L(w) = L'(w') = y$. Then the subgraph of G induced by w is a $2K_1$ (K_2) iff the subgraph of H induced by w' is a $2K_1$ (K_2 respectively). Hence the mapping Φ is an isomorph mapping $V(G) \to V(H)$. ■

3.3 *Two graphs G and H are isomorphic iff Y(G) equals Y(H).*

Proof: Straightforward from **3.2**. This theorem declares that $\Psi(G)$ is a complete invariant of graphs. ■

[10] In Section 6, we will define edge-labeling Λ and use lower Greek γ to denote the partial labeling of Λ.

4. Kelly-Ulam Conjecture

Around the Kelly-Ulam Conjecture, the collection of the vertex-deleted subgraphs of a graph G is called the deck of G as well as a vertex-deleted subgraph in the deck is called a card. Similarly the collection of the edge-deleted subgraphs of G is called the edge-deck of G. We will follow this convention. Now we introduce another Ψ numbers called unlabeled Ψ numbers, defined for unlabeled graphs and written $\Psi_U(G)$.

$\Psi_U(K_1) := 1$, $\Psi_U(2K_1) := 3$, $\Psi_U(K_2) := 5$, and

$$\Psi_U(G) := \prod_{i=1}^{r} P(i)^{\Psi(G - v_i)}\text{, where } r = |V| \text{ and } i < j \Leftrightarrow \Psi(G - v_i) \le \Psi(G - v_j).$$

An unlabeled graph G has only one unlabeled Ψ number Ψ_U. This formulation represents the condition of K-U directly. The term $\Psi(G - v_i)$ in the formula is the maximum labeled Ψ number of a vertex-deleted subgraph $G - v_i$ of G. We expand this definition to labeled graphs blending the concept of labeled Ψ numbers $\Psi(L)$ and unlabeled Ψ numbers $\Psi_U(G)$. We call such a Ψ numbering natural Ψ numbers and write $\Psi_N(L)$. At the top level of the Ψ-tree of a natural Ψ number $\Psi_N(L)$, it has an ordered set of labeled Ψ numbers $\Psi(G_L(i))$ of the vertex-deleted subgraphs. The rest part of the tree is similar to the ordinary labeled Ψ numbers.

$\Psi_N(K_1) := 1$, $\Psi_N(2K_1) := 3$, $\Psi_N(K_2) := 5$, and

$$\Psi_N(L) := \prod_{i=1}^{r} P(i)^{\Psi(G_L(i))}\text{, where } L \in L_N^G,\ r = |V| \text{ and } i < j \Leftrightarrow \Psi(G_L(i)) \le \Psi(G_L(j)).$$

$$\Psi_N(G) := \Psi_N(L_0) \ge \Psi_N(L) : {}^{\forall}L \in L_N^G\text{, where } L_N^G \text{ is the set of all labelings } L \text{ with valid } \Psi_N(L).$$

In this formulation every subgraphs of a labeled graph G_L share a common labeling L. Obviously a natural Ψ number $\Psi_N(L)$ is a labeled Ψ number $\Psi(L)$, therefore $\Psi_N(L) = \Psi(L)$. However all labelings $L \subseteq L^G$ do not necessarily give a $\Psi_N(L)$ due to the constraint condition at the top level of the Ψ-tree. Let L_N^G denote the set of all labelings L with a legitimate $\Psi_N(L)$ of G. Then the set L_N^G is a subset of L^G such as $L_N^G \subseteq L^G$. A graph G has at least one $\Psi_N(L)$, and $\Psi_N(G)$ is the maximum value among all $\Psi_N(L)$ of G. It is easy to see that $\Psi_U(G) \ge \Psi(G) \ge \Psi_N(G)$.

4.1 *Given two graphs G and H, if there exist labelings L of G and L' of H such that $Y_N(L)$ equals $Y_N(L')$, then G and H are isomorphic.*

Proof: The statement is true for graphs of order ≤ 2. Then we assume the order of graphs ≥ 3. Assume there exist labelings L of G and L' of H such as $\Psi_N(L) = \Psi_N(L')$. Then by the definition of natural Ψ numbers, for all $x \in \Sigma_2^n$, $\Psi_N(G_L(x))$ and $\Psi_N(H_{L'}(x))$ must be exactly coincident. Consider a mapping Φ: $V(G) \to V(H)$ decided by L and L' such that $\Phi = L'^{-1}L$. Let y be a number sequence in Σ_2^n such as $|y| = n - 2$. Take vertex pairs w in the graph G and w' in H mapped by L and L' respectively corresponding to y, where $L(w) = L'(w') = y$. Then it comes to be that the subgraph of G induced by w is a $2K_1$ (K_2) iff the subgraph of H induced by w' is a $2K_1$ (K_2 respectively). Hence the mapping Φ is an isomorph mapping $V(G) \to V(H)$, and G and H are isomorphic. ■

4.2 *If two graphs G and H are isomorphic, then for any labeling L of G, there exists a labeling L' of H such that $Y_N(L)$ equals $Y_N(L')$.*

Proof: The statement is trivially true for graphs of order ≤ 2. Then we assume the order of graphs ≥ 3. Assume that two graphs G and H are isomorphic. Then there is an isomorph mapping Φ: $V(G) \to V(H)$. Let an arbitrary labeling of G be L. We show that there is a labeling L' of H corresponding to L such that $\Psi_N(L) = \Psi_N(L')$. Consider a labeling L' such that $\Phi = \Phi L^{-1}L = L'^{-1}L = L'^{-1}I_nL$ and $L' = L\Phi^{-1}$. Let the Ψ-formulas of L and L' be Ψ and Ψ' respectively. For all number sequences $x \in \Sigma_2^n$, Φ maps x in Ψ to the same x in Ψ' through the identity I_n. Consequently every subgraph $G_L(x)$ in Ψ is mapped to the corresponding subgraph $H_{L'}(x)$ in Ψ' by the isomorph mapping Φ. Hence for all $x \in \Sigma_2^n$, $\Psi_N(G_L(x)) = \Psi_N(H_{L'}(x))$. This yields $\Psi_N(L) = \Psi_N(L')$. ■

4.3 *Two graphs G and H are isomorphic iff $\Psi_N(G)$ equals $\Psi_N(H)$.*

Proof: Since $\Psi_N(G)$ and $\Psi_N(H)$ are both natural Ψ numbers, if $\Psi_N(G) = \Psi_N(H)$, then by **4.1**, G and H are isomorphic. The converse. Without loss of generality we assume that G and H are isomorphic but $\Psi_N(G) > \Psi_N(H)$. Then by **4.2** there must be a labeling L' of H such that $\Psi_N(G) = \Psi_N(L') > \Psi_N(H)$. This conflicts the hypothesis that $\Psi_N(H)$ is the maximum natural Ψ number of H. Hence the statement is deduced to be true. Note that this theorem declares that $\Psi_N(G)$ is a complete invariant of graphs. ■

4.4 *Given two graphs G and H, $\Psi_N(G)$ equals $\Psi_N(H)$ iff $\Psi_U(G)$ equals $\Psi_U(H)$.*

Failed Proof: Assume $\Psi_N(G) = \Psi_N(H)$, then by **4.3**, $G \cong H$. This implies $\Psi_U(G) = \Psi_U(H)$. Next we prove the converse, i.e., "If $\Psi_U(G) = \Psi_U(H)$, then $\Psi_N(G) = \Psi_N(H)$" by applying both reductio ad absurdum and mathematical induction. Assume that there exists a natural number $k \geq 3$ such that for all graphs G and H of order $\leq k$, the proposition is true. Suppose graphs G and H of order $k + 1$, and assume $\Psi_U(G) = \Psi_U(H)$. Then by the definition of unlabeled Ψ numbers,

$$^{\forall}i \in \sigma_n, \Psi(G - u_i) = \Psi(H - v_i). \quad (1)$$

$$\Psi(G - u_1) \leq \Psi(G - u_2) \leq ... \leq \Psi(G - u_n), \quad (2)$$
$$\Psi(H - v_1) \leq \Psi(H - v_2) \leq ... \leq \Psi(H - v_n). \quad (3)$$

Let the labelings of G and H be L and L' respectively. The labeled graph of G and H labeled by the labelings L and L' are denoted G_L and $H_{L'}$ respectively. By the definition of natural Ψ numbers, for each number sequence x of Ψ-tree of the labeled graph G_L,

$$i < j \Leftrightarrow \Psi_N(G_L(i)) \leq \Psi_N(G_L(j)).$$

$$\Psi_N(G_L(1)) \leq \Psi_N(G_L(2)) \leq ... \leq \Psi_N(G_L(n)), \quad (4)$$
$$\Psi_N(H_{L'}(1)) \leq \Psi_N(H_{L'}(2)) \leq ... \leq \Psi_N(H_{L'}(n)). \quad (5)$$

Let Θ and Θ' be mappings: $\sigma_n \rightarrow \sigma_n$ which satisfy the following inequalities corresponding to (4) and (5) such as $\Theta(i) = j$, and let $G_U(\Theta(i))$ denote the unlabeled version of a labeled vertex-deleted subgraph $G_L(j)$.

$$\Psi_N(G_U(\Theta(1))) \leq \Psi_N(G_U(\Theta(2))) \leq ... \leq \Psi_N(G_U(\Theta(n))), \quad (6)$$
$$\Psi_N(H_U(\Theta'(1))) \leq \Psi_N(H_U(\Theta'(2)) \leq ... \leq \Psi_N(H_U(\Theta'(n)), \quad (7)$$

Note that the sets of those vertex-deleted subgraphs are 1-to-1 corresponding like,

$$\{G - u_i\} \leftrightarrow \{G_U(\Theta(i))\} \leftrightarrow \{G_L(i)\}, \text{ and } \{H - v_i\} \leftrightarrow \{H_U(\Theta'(i)\} \leftrightarrow \{H_{L'}(i)\}.$$

Vertex-deleted subgraphs $G - u_i$ and $G_U(\Theta(i))$ are unlabeled and the maximum natural Ψ number $\Psi_N(G)$ of an unlabeled graph G is unique. So for all $i \in \sigma_n$, there exists some number j such as $\Psi_N(G - u_i) = \Psi_N(G_U(\Theta(j)))$. Hence by the equation (1), the Ψ number sequences (6) and (7) must be coincident as well as (2) and (3) like the following.

$$^{\forall}i \in \sigma_n, \Psi_N(G_U(\Theta(i))) = \Psi_N(H_U(\Theta'(i)). \quad (8)$$

$\Psi_N(G_U(\Theta(i)))$ represents the maximum Ψ_N value of labeled subgraph $G_L(j)$. Further by **4.2**, and **4.3**,

$$\Psi_N(G_U(\Theta(i))) = \Psi_N(H_U(\Theta'(i))) \Leftrightarrow G_U(\Theta(i)) \cong H_U(\Theta'(i)) \Leftrightarrow G_L(j) \cong H_{L'}(j) \Leftrightarrow {}^{\forall}L, {}^{\exists}L': \Psi_N(G_L(j)) = \Psi_N(H_{L'}(j)).$$

Therefore we have,

$$^{\forall}L \in L_N^G, {}^{\exists}L' \in L_N^H, {}^{\forall}i \in \sigma_n: \Psi_N(G_L(i)) = \Psi_N(\mathrm{H}_{L'}(i)). \quad (A)$$

Assume $\Psi_N(G) \neq \Psi_N(H)$. Then by **4.3**, $G \not\cong H$. Hence by **4.1**, There is no pair of labelings L of G and L' of H such as $\Psi_N(L) = \Psi_N(L')$. Consequently by the definition of natural Ψ numbers,

$$^{\forall}L \in L_N^G, {}^{\forall}L' \in L_N^H, {}^{\exists}i \in \sigma_n: \Psi_N(G_L(i)) \neq \Psi_N(\mathrm{H}_{L'}(i)). \quad (B)$$

Obviously the equation (A) conflicts the inequality (B). By this contradiction, we deduce that the proposition, "if $\Psi_U(G) = \Psi_U(H)$, then $\Psi_N(G) = \Psi_N(H)$" is true for all graphs of order $k + 1$. Moreover it is easy to certify that the proposition is true for small graphs of order ≤ 3. This completes the mathematical induction. ■

Remark: Apparently our proof for **4.4** is incorrect as the assertion A is invalid. From **4.1**, **4.2** and **4.3**, we know

$$\Psi_N(L) = \Psi_N(L') \Leftrightarrow G \cong H \Leftrightarrow {}^{\forall}L, {}^{\exists}L': \Psi_N(G_L) = \Psi_N(H_{L'}) \Leftrightarrow \Psi_N(G) = \Psi_N(H).$$

However the above inference which derives assertion A does not work. Because $G_L(j)$ and $H_{L'}(j)$ are subgraphs and the theorems are not for subgraphs. Therefore the labelings of them must be considered independently. Let the independent labelings of $G_L(j)$ and $H_{L'}(j)$ be L_j and L'_j respectively. Then the equivalence should be correctly,

$$G_L(j) \cong H_{L'}(j) \Leftrightarrow {}^{\forall}L_j, {}^{\exists}L'_j: \Psi_N(G_{L_j}) = \Psi_N(H_{L'_j}).$$

We cannot derive the desired conclusion from the above equivalence. A vertex-deleted subgraph $G_L(i)$ is labeled by a labeling L of G, and L is so to say a total labeling which covers all the subgraphs of G but cannot cover every combinations of the local labelings L_j for the subgraphs. Now we will challenge the proof of **4.4** again. This time we try to prove it by showing that if there exists a pair of graphs of order k such as a counterexample of **4.4**, then there exists a pair of vertex-deleted subgraphs of them which is also the counterexample of it. Recall **4.4**.

4.4 *Given two graphs G and H,* $\Upsilon_N(G)$ *equals* $\Upsilon_N(H)$ *iff* $\Upsilon_U(G)$ *equals* $\Upsilon_U(H)$ *.*

Failed Proof: As the necessity part is self-evident, we prove the sufficiency part, i.e., "If $\Psi_U(G) = \Psi_U(H)$, then $\Psi_N(G) = \Psi_N(H)$". Suppose the smallest counterexample of the proposition such as a pair of graphs G and H of order k. Assume that $\Psi_U(G) = \Psi_U(H)$ but $\Psi_N(G) \neq \Psi_N(H)$. Let the labelings of G and H which give the maximum natural Ψ numbers $\Psi_N(G)$ and $\Psi_N(H)$ be L and L' respectively. By the assumption, for all $i \in \sigma_n$: $\Psi(G - u_i) = \Psi(H - v_i)$, and by the definition of natural Ψ numbers, there exists some $i \in \sigma_n$: such as $\Psi_N(G_L(i)) \neq \Psi_N(H_{L'}(i))$. Without loss of generality we assume that $L(u_i) = i$ and $L'(v_i) = i$. Let the vertex-deleted subgraphs $G - u_i$ and $H - v_i$ be G_i and H_i respectively. Then we have $\Psi_U(G_i) = \Psi_U(H_i)$ and $\Psi_N(G_i) \neq \Psi_N(H_i)$. The vertex-deleted subgraphs G_i and H_i is a counterexample of the proposition and their order is k - 1. This conflicts the hypothesis that G and H is the smallest counterexample. Therefore we deduce that the statement is true. ■

Remark: The flaw of the above proof is similar to the aforementioned proof. $\Psi_N(G_i) \neq \Psi_N(H_i)$ is true for the labelings L and L'. Of course we can borrow them as the partial labelings for the subgraphs G_i and H_i. The inequality is valid as far as the labelings L and L' work. Further if the number of the incompatible subgraphs pair is only one, then the labelings of G and H are enough to verify the discrepancy of the graphs. Hence the smallest counterexample must have two or more incompatible vertex-deleted subgraphs pairs. In this case subgraphs A and A' agree in a labeling pair L_a and L_a' and disagree in another labelings L_b and L_b', while subgraphs B and B' agree in labelings L_b and L_b'.

4.5 *Two graphs G and H are isomorphic iff their unlabeled* Υ *numbers are equal.*

Proof: Trivially true by **4.3** and **4.4**. This theorem declares that $\Psi_U(G)$ is a complete invariant of graphs. ■

4.6 Kelly-Ulam Conjecture: *Two graphs of order ³ 3 are isomorphic iff they have the same deck.*

Proof: The necessity part is self-evident. Then we prove the sufficiency part. Let two graphs of the same order n be G and H. The proposition to be proven is that if there is a mapping Φ: $V(G) \to V(H)$ such that

$${}^{\forall}v \in V(G),\ G - v \cong H - \Phi(v), \text{ then } G \cong H.$$

By the premise, ${}^{\forall}i \in \sigma_n$, $G - v_i \cong H - \Phi(v_i)$, then by **4.5**, it comes that ${}^{\forall}i \in \sigma_n$, $\Psi_U(G - v_i) = \Psi_U(H - \Phi(v_i))$. This yields $\Psi_U(G) = \Psi_U(H)$. Hence by **4.5** again, G and H are isomorphic. ■

4.7 Reconstruction Conjecture: *A graph of order ³ 3 is reconstructible from its deck.*

Proof: Let the deck of a graph G be D. It is obvious that as far as the given deck D is obtained legitimately, there must be at least one graph H whose deck is D. Furthermore it is also sure that we can find such a graph H by some exhaustive enumeration in a finite time. The problem here is whether the graph H obtained in such a way is actually unique or not. And **4.6** guarantees this. ■

5. K-U Counterexamples

As was remarked at the second proof of **4.4**, a pair of graphs of the smallest counterexample must have at least two incompatible subgraphs. Our difficulty is that we have no ground to assert that one of them is surely a non-isomorphic subgraphs pair as we are disable to check all the cases of exhaustive combinations of the labelings of the subgraphs by verifying the labelings of the parent graphs. Simultaneously we know that our Ψ numbers are too redundant to determine the isomorphism of the graphs. In fact a Ψ-tree of a Ψ number has

$$\sum_{k=0}^{n-2}\frac{n!}{(n-k)!}=\sum_{k=0}^{n-2}\binom{n}{k}k!$$

nodes. Obviously the information is more than abundant. To what extent can we reduce the structure of Ψ numbering system? We will show that we can simplify it up to just one vertex-deleted subgraph and a small attachment part of it. Hereon the attachment part is the subgraph of G consists of the deleted vertex v and other vertices than v of G and the edges connecting v to other vertices. We introduce a newly defined Ψ numbers and call it supernatural Ψ numbers. Ψ_S denotes such a supernatural Ψ number.

$\Psi_S(K_1) := 1$, $\Psi_S(2K_1) := 3$, $\Psi_S(K_2) := 5$, and

$\Psi_S(L,v) := F_0(L,v) \times F_1(L,v)$, where $L \in \mathcal{L}^G$ and $v \in V$.

$F_0(L,v) := P(L(v))^{\Psi(G_L(L(v)))}$, where G_L is a labeled graph of G labeled by L, and

$F_1(L,v) := \prod_{i=1}^{r} P(i)^{f_L(L(v),i)}$, where $r = |V|$, $L(v) \in \sigma_n$ is the label number of the vertex v.

$$f_L(l,i) := \begin{cases} 0 & (i = l) \\ 0 & (i \neq l, L^{-1}(i)L^{-1}(l) \notin E) \\ 1 & (i \neq l, L^{-1}(i)L^{-1}(l) \in E) \end{cases}.$$

$\Psi_S(L) := \Psi_S(L,v_0) \geq \Psi_S(L,v))$: $^{\forall}v \in V$, where $L \in \mathcal{L}^G$.

$\Psi_S(G) := \Psi_S(L_0) \geq \Psi_S(L)$: $^{\forall}L \in \mathcal{L}^G$, where $\mathcal{L}^G$ is the set of all labelings L of G.

$\Psi_S(L,v)$ is a function $(\mathcal{L}^G \times V) \to N$ such as a product $F_0 \times F_1$, where L is a labeling of G, $v \in V$, and N is the infinite set of natural numbers. The function $F_0(L,v)$ represents the vertex-deleted subgraph labeled by L, and v is the vertex deleted from the labeled graph G_L. Note that $\Psi(G_L(L(v)))$ is a labeled Ψ number but neither a supernatural Ψ number nor a natural Ψ number. The function $F_1(L,v)$ represents the attachment part of the vertex v connecting to the vertex-deleted subgraph $G_L(L(v))$, and evaluates the adjacency of the vertex v with other vertices of G. Supernatural Ψ number $\Psi_S(L,v)$ is a kind of labeled Ψ number and determines a unique Ψ number for each labeling L and vertex v.

5.1 *Given two graphs G and H, if there exist a labeling L and a vertex u of G, as well a labeling L' and a vertex v of H such that* $\mathrm{Y}_S(L,u)$ *equals* $\mathrm{Y}_S(H,v)$*, then G and H are isomorphic.*

Proof: The statement is provable almost similarly to the aforementioned proofs. The statement is true for graphs of order ≤ 2. Then assume the order of graphs ≥ 3. Suppose that there exist a labeling L and a vertex u of G, and a labeling L' and a vertex v of H such as $\Psi_S(L,u) = \Psi_S(L',v) = F_0(L,u) \times F_1(L,u) = F_0(L',v) \times F_1(L',v)$. Then by the definition of supernatural Ψ numbers, $F_0(L,u) = F_0(L',v)$ and $F_1(L,u) = F_1(L',v)$. Consequently

$$\Psi(G_L(L(u))) = \Psi(H_{L'}(L'(v))), \tag{1}$$

$$^{\forall}i \in \sigma_n \; f_L(L(u),i) = f_{L'}(L'(v),i). \tag{2}$$

Consider a mapping Φ: $V(G) \to V(H)$ decided by L and L' such that $\Phi = L'^{-1}L$. At first from the definition of supernatural Ψ numbers, it must be $L(u) = L'(v)$. Suppose ordered vertex pairs w and w' in the graphs G and H respectively such as $L(w) = L'(w') = (x,y)$, where $x,y \in \sigma_n$ and $x, y \neq L(u) = L'(v)$. Then by the equation (1) and the definition of labeled Ψ numbers, $\Psi(G_L(\sigma_n - (x,y))) = \Psi(H_{L'}(\sigma_n - (x,y)))$. Then it comes to be that the subgraph of G induced by w is a $2K_1$ (K_2) iff the subgraph of H induced by w' is a $2K_1$ (K_2 respectively). Now turn our eyes to the attachment parts. Suppose $L(u) = L'(v) = x$ and $y \in \sigma_n$, then from the equation (2), $f_L(x,y) = f_{L'}(x,y)$. Consequently for all vertex pairs w and w' in the attachment part of respectively G and H such as $L(w) = L'(w') = (x,y)$, the subgraph of G induced by w is a $2K_1$ (K_2) iff the subgraph of H induced by w' is a $2K_1$ (K_2 respectively). Thus the mapping Φ is an isomorph mapping $V(G) \to V(H)$, and G and H are isomorphic. ■

5.2 *If two graphs G and H are isomorphic, then for any labeling L and a vertex u of G, there exist a labeling L' and a vertex v of H such that* $Y_S(L,u)$ *equals* $Y_S(L',v)$.

Proof: The proof is almost similar to the aforementioned proofs. The statement is true for graphs of order ≤ 2. Then assume that the order of graphs ≥ 3 and two graphs G and H are isomorphic. Then there is an isomorph mapping Φ: $V(G) \rightarrow V(H)$. Let an arbitrary labeling of G be L and a vertex of G be u. We show that there is a labeling L' and a vertex v of H corresponding to L and u such as $\Psi_S(L,u) = \Psi_S(L',v)$. Suppose a labeling L' such that $\Phi = \Phi L^{-1}L = L'^{-1}L = L'^{-1}I_nL$ and $L' = L\Phi^{-1}$. The mapping Φ maps every subgraphs $G_L(x)$ in the vertex-deleted subgraph $G_L(L(u))$ to the corresponding subgraphs $H_{L'}(x)$ in the vertex-deleted subgraph $H_{L'}(L'(v))$. This yields $\Psi_S(G_L(L(u))) = \Psi_S(H_{L'}(L'(v))) = F_0(L,u) = F_0(L',v)$ and $L(u) = L(v)$. Next turn our eyes to the attachment parts. Suppose $L(u) = L(v) = x$ and $y \in \sigma_n$. Φ maps every vertex pairs (x,y) in the attachment part of G_L to the corresponding vertex pairs (x,y) in the attachment part of $H_{L'}$. Then for all $y \in \sigma_n, f_L(x,y) = f_{L'}(x,y)$, and this yields $F_1(L,u) = F_1(L',v)$. Hence $F_0(L,u) \times F_1(L,u) = F_0(L',v) \times F_1(L',v) = \Psi_S(L,u) = \Psi_S(L',v)$. ∎

5.3 *Two graphs G and H are isomorphic iff* $Y_S(G)$ *equals* $Y_S(H)$.

Proof: Abbreviated as it is quite same with the proof of **4.3**. ∎

5.4 *Two graphs G and H of order n are isomorphic iff for all i £ n,* $Y(G - u_i)$ *equals* $Y(H - v_i)$.

Counterexample: Proposition **5.4** is a paraphrase of K-U Conjecture. We admit that we had better consider the possibility of the existence of K-U counterexamples. Let a labeling of G be L and a vertex of G be u, as well a labeling of H be L' and a vertex of H be v. From **5.1**, **5.2** and **5.3**,

$$\Psi_S(L,u) = \Psi_S(L',v) \Leftrightarrow G \cong H \Leftrightarrow {}^{\forall}L, {}^{\forall}u, {}^{\exists}L', {}^{\exists}v: \Psi_S(L,u) = \Psi_S(L',v) \Leftrightarrow \Psi_S(G) = \Psi_S(H).$$

Assume for all $i \in \sigma_n$ $\Psi(G - u_i) = \Psi(H - v_i)$. Then by the definition of supernatural Ψ numbers,

$$ {}^{\forall}i, {}^{\exists}L, {}^{\exists}L': \ F_0(L,u_i) = F_0(L',v_i). \tag{1}$$

The minimum condition for G and H to be isomorphic:

$$ {}^{\exists}L, {}^{\exists}L', {}^{\exists}u. {}^{\exists}v: \ F_0(L,u) \times F_1(L,u) = F_0(L',v) \times F_1(L',v). \tag{2}$$

For the counterexample, take the negation of (2),

$$ {}^{\forall}L, {}^{\forall}L', {}^{\forall}u. {}^{\forall}v: \ F_0(L,u) \neq F_0(L',v) \vee F_1(L,u) \neq F_1(L',v). \tag{3}$$

From (1) and (3) it must be

$$ {}^{\forall}i, {}^{\exists}L, {}^{\exists}L': F_0(L,u_i) = F_0(L',v_i) \Rightarrow F_1(L,u_i) \neq F_1(L',v_i). \tag{4}$$

We will call such a situation as (4) F1-friction. It is a very invisible contradiction located at the attachment parts. Now we know that any K-U counterexample must have F1-frictions. In a counterexample, for every isomorph mappings between corresponding vertex-deleted subgraphs, there exists a F1-friction relevant to the mapping.

Counter-counterexample: We want to show the existence of a counterexample for K-U counterexamples. However this is somewhat of a paradoxical attempt, because (1) if K-U is true, then a K-U counterexample does not exist in the first place, hence it is impossible to construct a counter-counterexample. (2) if K-U is not true, then a K-U counterexample is constructible, but a counter-counterexample is not. Hence even if it is able, it is inevitable to be pure theoretical one. What we are seeking is a K-U counterexample which has a non-isomorphic subgraphs pair. The ground of our theory is that it may be probable that the F1-friction in a counterexample is remaining in some subgraphs pair. If ${}^{\forall}L, {}^{\forall}L'$: $F_1(L,u) = F_1(L',v)$, it becomes the simplest counter-counterexample. If either u (respectively v) is isolated or u (v) is adjacent with all other vertices, it will happen. However we already know that disconnected graphs cannot yield a counterexample. Alternatively suppose a K-U counterexample pair G and G' of order k. Can't we find a graph pair H and H' of order $k + 1$ preserving the F1-frictions of G and G'. If there is, it is a counter-counterexample. But this does not happen because every vertex-deleted subgraphs of a counterexample must be pairwise isomorphic and G and G' are not isomorphic. So we cannot help concluding that it is possible that a K-U counterexample exists. Any K-U counterexample never shares a pair of counterexample subgraphs inside and all of their vertex-deleted subgraphs are so to say pairwise incompatible, i.e., having their F1-frictions.

6. Edge-Reconstruction Conjecture

Our final stage to investigate the K-U Conjecture should be to examine the possibility of edge-reconstruction. Because we already reached the conclusion that K-U Conjecture may not hold, and if edge-reconstruction is possible, we would expect to discover a K-U counterexample whose Ψ numbers are equal but edge Ψ numbers are different. On the contrary if the edge-reconstruction is not possible, i.e., if there exists a counterexample for edge-reconstruction, then it turns out to be also a counterexample of K-U. Recall the conjecture posed by Harary [14].

Edge-Reconstruction Conjecture: *A graph G of size ³ 4 is reconstructible from its edge-deleted subgraphs.*

An edge-deleted subgraph G - e is the subgraph of G obtained by removing an edge e from G. As well a k-edge-deleted subgraph G - S is defined similarly, where $S \subset E$ and $|S| = k$. Suppose two isomorphic graphs G and H, and an isomorph mapping Φ: $V(G) \to V(H)$. Then there exists a mapping Θ: $E(G) \to E(H)$ such that $^{\forall}e \in E(G)$: $\Phi(e) = e$' $\Leftrightarrow$ $\Theta(e) = e$'. We call such a mapping Θ an edge-isomorph mapping from $E(G)$ onto $E(H)$.

We define edge-labelings of a graph G with m edges as follows. Λ denotes a mapping from the edge set E onto the number sequence $\sigma_m = (1,2,...,m)$ such as $\Lambda(e_i) = j \in \sigma_m$ and $e_i \in E$. Λ is called an edge-labeling of G and the number j is called the edge-label of the edge e_i. We call the range σ_m associated with the edge-labeling Λ the label set of Λ. Λ^G denotes the set of all edge-labelings Λ of G, and G_Λ denotes the edge-labeled graph of G labeled by an edge-labeling Λ. An unlabeled graph G of size m has m! edge-labelings. A partial labeling λ of edge-labeling Λ of G is a mapping $E(G_\lambda) \to \sigma \subseteq \sigma_m$ such as $^{\forall}e \in E(G_\lambda)$: $\Lambda(e) = \lambda(e)$, where G_λ is a subgraph of an edge-labeled graph G_Λ, and σ is the sub-sequence of $\sigma_m = (1,2,...,m)$ of length $|E(G_\lambda)|$. We call the range σ of the partial labeling λ the label set of λ.

Suppose a subgraph G_λ labeled by a partial labeling λ of an edge-labeled graph G_Λ labeled by the edge-labeling Λ. Σ^m is the set of all sequences of distinct natural numbers $i \in \sigma_m$ of length $\leq m$. Let σ be the label set of λ and x be an arbitrary sequence of distinct natural numbers $i \in \sigma \subseteq \sigma_m$. $\Sigma^s \subseteq \Sigma^m$ denotes the set of all such number sequences x. Σ_k^s denotes a subset of Σ^s such as $\Sigma_k^s = \{x \mid x \in \Sigma^s, |x| \leq |\sigma| - k\}$, where $k \leq |\sigma|$ and $\Sigma_k^s \subseteq \Sigma^s$. The complementary number sequence σ - x of a number sequence x is denoted $\overline{x} = \sigma - x$, and $\overline{x}(i)$ denotes the i-th number in the number sequence $\overline{x}$. Suppose a number sequence $x \in \Sigma_3^s$ and the edge subset $S \subset E(G_\lambda)$ such as $\Lambda(S) = x$, then $G_\lambda(x)$ denotes k-edge-deleted subgraph G_λ - S, where $k = |x|$. We introduce a new Ψ number called edge-connected Ψ numbers and written Ψ_C. The edge-connected Ψ number Ψ_C is defined for connected and edge-labeled graphs.

$\Psi_C(pK_1) := 2^p$, $\Psi_C(pK_1+K_2) := 2^p\times 3$, $\Psi_C(pK_1+2K_2) := 2^p\times 5$, $\Psi_C(pK_1+P_2) := 2^p\times 7$, $\Psi_C(pK_1+3K_2) := 2^p\times 11$,
$\Psi_C(pK_1+K_2+P_2) := 2^p\times 13$, $\Psi_C(pK_1+P_3) := 2^p\times 17$, $\Psi_C(pK_1+K_{1,3}) := 2^p\times 19$, and $\Psi_C(pK_1+K_3) := 2^p\times 23$,
where $p = 0,1,2$ is the number of the isolated vertices in the graph.

$\Psi_C(G_l(x)) := 2^p \prod_{i=1}^{r} P(i+1)^{\Psi_C(G_l(x+\overline{x}(i)))}$, where $x \in \Sigma_3^s$, $r = |\overline{x}| \geq 4$, $\overline{x} = \sigma - x$, σ is the label set of λ,
$p = 0,1,2$ is the number of the isolated vertices in $G_\lambda(x)$, and
$G_l(x+\overline{x}(i))$ is the edge-deleted subgraph $G_\lambda(x) - \lambda^{-1}(\overline{x}(i))$.

$\Psi_C(G_l) := \Psi_C(G_l(f))$, where G_λ is an edge-labeled graph labeled by partial labeling λ of edge-labeling Λ.
$\Psi_C(\Lambda) := \Psi_C(G_\Lambda)$, where $\Lambda \in \Lambda^G$, and G_Λ is an edge-labeled graph of a connected graph G labeled by Λ.
$\Psi_C(G) := \Psi_C(\Lambda_0) \geq \Psi_C(\Lambda) : {}^{\forall}\Lambda \in \Lambda^G$, where G is an unlabeled connected graph.

G_λ is an edge-labeled graph labeled by a partial labeling λ of an edge-labeling Λ. In general, G_λ is a subgraph of edge-labeled graph G_Λ labeled by the edge-labeling Λ. If the labeling λ is total, i.e., $G_\lambda = G_\Lambda$ then $\Psi_C(G_\lambda)$ turns to $\Psi_C(\Lambda)$ which represents the Ψ number value of the edge-labeled graph G_Λ. $\Psi_C(G)$ represents the maximum Ψ_C value among all of the $\Psi_C(\Lambda)$. The $\Psi_C(G)$ is defined for connected and unlabeled graphs. $G_\lambda(x)$ denotes the k-edge-deleted subgraph of G_λ, and x is the number sequence of the edge-labels of the deleted edges.

A graph $G(V,E)$ can be said to consist of isolated vertices $V_Y \subseteq V$ and the maximum edge-induced subgraph $G_X(V - V_Y, E)$. In the formula of $\Psi_C(G_\lambda(x))$ the first term 2^p is for the isolated vertices $V_Y(G_\lambda(x))$ in the graph, and the right hand of the prime number $P(i)$ represents the maximum edge-induced subgraph of $G_\lambda(x)$. Since the graph G_λ is assumed to be connected, the root node of the Ψ-tree does not contain any isolated vertices in the first place. On the other hand, other nodes than the root possibly contain at most 2 isolated vertices. The Ψ-formula $\Psi_C(G_\lambda(x))$ is defined for graphs with 4 or more edges, while the Ψ numbers of the smaller graphs of size ≤ 3 are predetermined as above.

The edge-connected Ψ numbers $\Psi_C(G_\lambda(x))$ of k-edge-deleted subgraphs $G_\lambda(x)$ is defined recursively for all number sequences $x \in \Sigma_3^s$ and $\Psi_C(G_\lambda)$ is the case when x is the empty set ϕ, where σ is the label set of the partial labeling λ. The computation of a $\Psi_C(G_\lambda)$ forms a tree such that its nodes are Ψ numbers $\Psi_C(G_\lambda(x))$ in some computation level $0 \le i \le |\sigma| - 3$. Every node in level i is an edge-deleted subgraph of its parent node, and the number of edges (excluding the edge incident with the parent node) of an inner node is $|\sigma| - i$. Every leaves are the union of one of five small graphs of size 3 with prescribed Ψ_C numbers and p isolated vertices, where $0 \le p \le 2$. The number sequence $x \in \Sigma_3^s$ represents the set of deleted edges in the upper levels in that order, and works as an indicator on the Ψ-tree, while $\overline{x}$ represents the remained edges in $G_\lambda(x)$. Note that the edge-labeled graph G_λ is assumed to be connected but the subgraphs $G_\lambda(x)$ is not necessarily connected.

To abbreviate the description we introduce some auxiliary notations. Let e_1 and e_2 be edges. We write e_1-e_2 when e_1 and e_2 are adjacent as well as we write $e_1 \ldots e_2$ when e_1 and e_2 are not adjacent. $G(T)$ denotes the edge-induced subgraph of G induced by edge subset $T \subseteq E(G)$.

6.1 *Given two connected graphs G and H of size ³ 5, and a mapping Q: E(G) ® E(H). If for all edge-triples T of G, G(T) is isomorphic to H(Q(T)), then G and H are isomorphic and the mapping Q is an edge-isomorph mapping.*

Proof: Assume that there exists a mapping Θ: $E(G) \to E(H)$ such that for all edge-triples T of G, edge-induced subgraphs $G(T)$ and $H(\Theta(T))$ are isomorphic. Let an arbitrary labeling of G be L. We will transcribe the labeling L to a labeling L' of H and show that L and L' forms an isomorph mapping. There are 5 kinds of edge-triple graphs, $3K_2$, K_2+P_2, P_3, $K_{1,3}$, and K_3. If G and H have no $K_{1,3}$, then graphs are either P_{n-1} or C_n. In this case it is easy to certify that G and H are isomorphic. Hence assume that G has a $K_{1,3}$. Let the center vertex of the $K_{1,3}$ be x and the label of x be l, where $L(x) = l$. Then there exists a corresponding $K_{1,3}$ in H and we can determine the label of the center vertex x' of the $K_{1,3}$ in H like $L'(x') = l$. Secondly there exist edges e of G and $e' = \Theta(e)$ of H such as $e = (x,u)$ and $e' = (x',v)$. Since the labels of the vertices x, u and x' are predetermined, we can decide the label of the vertex v in H like $L'(v) = L(u)$. As the graphs are connected, we can decide all of labels in H by repeating the similar operation and eventually obtain the total labeling L' of H. However it is probable that there happen some conflicts on this labeling operation.

Let $T = (x,y,z)$ be an edge-triple in G and the corresponding edge triple in H be $T' = (x',y',z')$, where $\Theta(x) = x'$, $\Theta(y) = y'$, and $\Theta(z) = z'$. Suppose that a label conflict happened at the edges z and z'. Then we may infer that x and y are incident with different vertices of z, on the other hand x' and y' are incident with a same vertex of z', and vice versa. However in this case $G(T)$ is a P_3 or a K_3, and $H(T')$ is a $K_{1,3}$, hence $G(T)$ is not isomorphic to $H(T')$. Therefore such cases are excluded by the premise. Most possible conflicts are excluded by this reason. Now consider the case when both $G(T)$ and $H(T')$ are P_3 but the edge sequences are not coincident such as x-y-z and y'-x'-z', where $T=(x,y,z)$, and $T' = (x',y',z')$. We will show that by the premise, such a discrepancy does not happen. As the size of graphs is larger than 4, there exist another edge-triples (y,z,a) and (y',z',a'), where $\Theta(a) = a'$. Since y' and z' are not adjacent but y and z are adjacent, a' must be adjacent to either y' or z'.

Assume a' is adjacent to z', then (y',z',a') becomes K_2+P_2. To let (y,z,a) be K_2+P_2, a must be non-adjacent to both y and z. Note that a' cannot be adjacent to x' because otherwise (x',z',a') forms either $K_{1,3}$ or K_3, and (x,z,a) can be neither of them as x and z are not adjacent. In this formulation (x',z',a') forms a P_3. To let (x,z,a) be P_3, a must be adjacent to both x and z. This makes (x,y,z,a) a C_4 and it requires a' to be connected with y'. Obviously the discrepancy between $C_4(x,y,z,a)$ and $C_4(y',x',z',a')$ cannot be detected by the edge triples on the C_4. However as the size of graphs is larger than 4, we have the fifth edges b and $b' = \Theta(b)$ in the graphs. Without loss of generality we assume that b and b' are not adjacent to any other four edges in the C_4. Then edge-triple (b,x,z) is a $3K_2$ and (b',x',z') is a K_2+P_2. Hence a contradiction. Note that it is easy to certify that even if b and b' are adjacent to the edges in the C_4, a similar argument is valid. That is to say, a disagreement on a pair of C_4 indicates a contradiction anytime.

Now assume a' is adjacent to y' then (y',z',a') becomes a K_2+P_2. To make (y,z,a) a K_2+P_2, a must be non-adjacent to both y and z. However in this case, (x,y,a) forms a K_2+P_2 and (x',y',a') forms one of P_3, $K_{1,3}$, and K_3. Thus a contradiction again. Assume a' is adjacent to both y' and z', then (y',x',z',a') becomes a C_4. Such a case is already simulated at the previous paragraph. Thus in any case accompanied with discrepant edge sequences of P_3 like x-y-z and y'-x'-z', we encounter a contradiction to the premise. In other words, no discrepancy exists in the edge sequences of pair of corresponding P_3 under the given premise.

Since $3K_2$ and K_2+P_2 have at most two neighboring edges, they do not cause any conflicts in the labeling. $K_{1,3}$ and K_3 have three edges mutually adjacent but the edge-adjacency in them are determined definitely regardless the order of the edges. Thus P_3 is the only one case which has the probability of labeling conflicts. However we already confirmed that every pair of P_3 always coincide. Therefore we can transcribe the labeling L of G to the labeling L' of H with no conflicts as far as all of edge-triples are pairwise isomorphic. Suppose a mapping Φ: $V(G) \to V(H)$ such as $\Phi = L'^{-1}L$. As we assigned the same label to the corresponding vertices, without loss of generality for all $i \in \sigma_n$, $L(u_i) = L'(v_i) = i$ and $\Phi(u_i) = L'^{-1}L(u_i) = v_i$. Then for all edges $e = (u_i,u_j) \in E(G)$, $\Theta(e) = \Phi(e) = \Phi(u_i,u_j) = \Phi(u_i)\Phi(u_j) = v_iv_j = e' \in E(H)$. Hence G and H are isomorphic and the mapping Θ is an edge-isomorph mapping. ■

Remark: $2P_2$ is a disconnected graph with 6 vertices and 4 edges. Lemma **6.1** does not hold for $2P_2$. Say G=(1,2)+ (3,4) and H = (1,4) + (2,3), where {1,...,4} are edges of the graphs and (i,j) is a P_2 with adjacent edges $\{i,j\}$. Obviously for all $i \in$ {1,...,4}, edge-triple subgraphs $G - i$ and $H - i$ are isomorphic but the mapping Θ: (1,2,3,4) $\to$ (1,2,3,4) does not give the edge-isomorphism. P_4 is a connected graph with 5 vertices and 4 edges. Lemma **6.1** does not hold for this graph too. P_4 has 2 P_3 and 2 K_2+P_2 as its edge-deleted subgraphs. Two P_4 with edge sequences 1-2-3-4 and 4-2-3-1 have isomorphic edge-triple subgraphs, but the mapping Θ: (1,2,3,4) $\to$ (1,2,3,4) is not an edge-isomorph mapping. As was described above, this situation is same for C_4 case. That is, the lemma does not hold for graphs of order ≤ 4.

The line graph $L(G)$ of G is a graph whose vertex is an edge of G and two vertices i and j are adjacent iff edges i and j are adjacent in G. If G and H are isomorphic, then obviously $L(G)$ and $L(H)$ are isomorphic. For the converse, Hassler Whitney proved the following theorem in 1932 [40]. We will give another proof of **6.1** using this theorem .

Whitney's Theorem: *Let G and H be connected graphs with isomorphic line graphs. Then G and H are isomorphic unless one is K_3 and the other is $K_{1,3}$.*

Alternative proof for 6.1: Assume that there exists a mapping Θ: $E(G) \to E(H)$ such that for all edge-triples T of G, edge-induced subgraphs $G(T)$ and $H(\Theta(T))$ are isomorphic. If the adjacency of all edges of the graphs are known, then we can compose the line graphs $L(G)$ and $L(H)$ of G and H respectively, and prove the statement easily by applying Whitney's theorem. There are 5 kinds of edge-triple graphs, $3K_2$, K_2+P_2, P_3, $K_{1,3}$, and K_3. Among them $3K_2$, $K_{1,3}$ and K_3 give us complete information of the edge-adjacency of the edge-triples, whereas K_2+P_2 and P_3 do not.

Let an edge-triple of a P_3 be T_1 and the other P_3 be T_2. If the intersection $T_1 \cap T_2 = \{e_1,e_2\}$, then edges e_1 and e_2 are adjacent in almost cases. While if e_1 and e_2 are in a C_4, then e_1 and e_2 may not be adjacent. We can check if the edges e_1 and e_2 are contained in a C_4 or not in some exhaustive way. However as was mentioned in the previous proof, the adjacency of e_1 and e_2 cannot de determined by the edge-triples in the C_4 alone. Fortunately the size of graphs is larger than 4 and we have always the fifth edge e_5. There are three cases, (1) e_5 is not adjacent to any edges in the C_4. (2) a vertex of e_5 is incident with a vertex on the C_4, (3) two vertices of e_5 is on the C_4.

In case (1) it is easy to certify that e_1 and e_2 are not adjacent. Because in this case the edge-triple (e_1,e_2,e_5) forms a $3K_2$. And this means that the edges are mutually not adjacent. Let us simulate the case (2). Let the C_4 be e_1-e_2-e_3-e_4-e_1 and assume e_5 and the adjacent edges e_1-e_2 forms a $K_{1,3}$. Then from $K_{1,3}(e_1,e_2,e_5)$, we know e_1-e_2, e_2-e_5, e_1-e_5, and from e_1-e_5 and K_2+P_2 (e_1,e_3,e_5), we know $e_1 \ldots e_3$, $e_3 \ldots e_5$. Further from e_2-e_5 and K_2+P_2 (e_2,e_4,e_5), we know $e_2 \ldots e_4$, $e_4 \ldots e_5$, and from e_1-e_2, $e_1 \ldots e_3$, and $P_3(e_1,e_2,e_3)$, we know e_2-e_3. As well from e_1-e_2, $e_2 \ldots e_4$, and $P_3(e_1,e_2,e_4)$, we know e_1-e_4, and from e_2-e_3, $e_2 \ldots e_4$, and $P_3(e_2,e_3,e_4)$, we know e_3-e_4. Thus we get the full information of the adjacency with respect to the five edges (e_1,e_2,e_3,e_4,e_5). In the similar way the case (3) can be certified easily. Consequently if there exist at least two P_3 which contain an edge pair (e_i,e_j) in common, then we can definitely decide the adjacency of the edge pair (e_i,e_j). Since graphs are connected and the size is larger than 4, every adjacent edge pair (e_i,e_j) of the graphs is in at least one P_3.

Suppose an edge pair (e_1,e_2) which is in $P_3(e_1,e_2,e_3)$, but not contained in other P_3. Without loss of generality we assume that the $P_3(e_1,e_2,e_3)$ is in the form e_1-e_2-e_3. Further we assume that e_1 is adjacent to no other edges e than e_2. Because otherwise the edge-adjacency is collectively decided by $K_{1,3}(e_1,e_2,e)$, or the hypothesis "(e_1,e_2) is in only one P_3" shall be broken by $P_3(e,e_1,e_2)$. Adding to it we assume that e_2 is adjacent to no other edges than e_1 and e_3 by the same reason. Then there exists $P_3(e_2,e_3,e_4)$ and (e_1,e_2,e_3,e_4) forms a P_4: e_1-e_2-e_3-e_4 for there is no other edges than e_3 which have additional adjacent edges. By the premise the graphs have more than 4 edges, and there exists the fifth edge e_5 in the graph. It is enough to consider the following two cases. One is the case where (e_1,e_2,e_3,e_4,e_5) forms a P_5: e_1-e_2-e_3-e_4-e_5. And the other is the case where e_5 is connected to e_3 and (e_3,e_4,e_5) forms a $K_{1,3}$. In either case it is no hard to show that the edge-triple subgraphs of the edge-quintuple (e_1,e_2,e_3,e_4,e_5) determines the adjacency of all edges. The former case is much easier. For the latter case, $K_{1,3}(e_3,e_4,e_5)$ determines the connections, e_3-e_4, e_4-e_5, and e_3-e_5. From e_4-e_5 and $K_2+P_2(e_1,e_4,e_5)$, we know $e_1\ldots e_4$, $e_1\ldots e_5$. From e_4-e_5 and $K_2+P_2(e_2,e_4,e_5)$, we know $e_2\ldots e_4$, $e_2\ldots e_5$. As well from e_3-e_4 and $K_2+P_2(e_1,e_3,e_4)$, we know $e_1\ldots e_3$, then from $e_1\ldots e_3$, and P_3 (e_1,e_2,e_3), we obtain e_1-e_2, e_2-e_3.

Thus we can decide all of the edge-connections in G from the set of edge-triple subgraphs of G and compose the line graph $L(G)$ of G. Since there is a mapping $\Theta: E(G) \to E(H)$ and for all edge-triples T of G, $G(T) \cong H(\Theta(T))$, the adjacency of the edges are exactly same for the two graphs. Therefore it is sure that we can compose the line graph $L(H)$ of H as well as $L(G)$, and $L(G)$ is isomorphic to $L(H)$, while Θ turns to an isomorph mapping $V(L(G)) \to V(L(H))$. Consequently by Whitney's theorem, G and H are isomorphic and the mapping Θ is an edge-isomorph mapping. The exception of $K_{1,3}$ and K_3 case in Whitney's theorem is excluded by the size boundary of the premise. ■

Remark: It is easy to show that (1) **Whitney's theorem** and (2) lemma **6.1** are equivalent for all graphs of size ≥ 5. If the line graphs $L(G)$ and $L(H)$ are isomorphic, then there exists an isomorph mapping Θ: $V(L(G)) \to V(L(H))$ and for all vertex-triples S of $L(G)$, the induced subgraphs of $L(G)$ and $L(H)$ respectively induced by S and $\Theta(S)$ are isomorphic. Obviously Θ is also an edge-isomorph mapping: $E(G) \to E(H)$ and for all edge-triples T of G, $G(T) \cong H(\Theta(T))$. Therefore (2) ⇒ (1). For the converse, the proof described above shows that if there exists an mapping Θ: $E(G) \to E(H)$ and for all edge-triples T of G, $G(T) \cong H(\Theta(T))$, then $L(G) \cong L(H)$. Hence (1) ⇒ (2).

6.2 *Given two connected graphs G and H, if there exist labelings* L *of G and* L' *of H such that* Y_C(L) *equals* Y_C(L'), *then G and H are isomorphic.*

Proof: It is easy to see that the statement is true for graphs of size ≤ 4. Then we assume that the size of graphs ≥ 5 and there exist edge-labelings Λ of G and Λ' of H such as $\Psi_C(\Lambda) = \Psi_C(\Lambda')$. By the definition of edge-connected Ψ numbers, for all $x \in \Sigma_3^m$, $\Psi_C(G_\Lambda(x))$ and $\Psi_C(H_{\Lambda'}(x))$ must be exactly coincident. Let y be a number sequence in Σ_3^m such as $|y| = m - 3$. Take ordered edge-triples $w = (e_i,e_j,e_k)$ in the graph G and $w' = (e'_i,e'_j,e'_k)$ in H mapped by Λ and Λ' respectively corresponding to y, where $\Lambda(w) = \Lambda'(w') = y$. Then it comes to be that the subgraph of G induced by w is isomorphic to the subgraph of H induced by w'. Suppose a mapping Θ: $E(G) \to E(H)$ such as $\Theta = \Lambda'^{-1}\Lambda$. Then the mapping Θ apparently satisfies the condition of the lemma **6.1**, therefore Θ is an edge-isomorph mapping $E(G) \to E(H)$ and G and H are isomorphic. ■

6.3 *If two connected graphs G and H are isomorphic, then for any edge-labeling* L *of G, there exists an edge-labeling* L' *of H such that* Y_C(L) *equals* Y_C(L').

Proof: The statement is trivially true for graphs of size ≤ 3. Then we assume that the size of graphs ≥ 4. Assume that two graphs G and H are isomorphic. Then there exists an isomorph mapping Φ: $V(G) \to V(H)$ and an edge-isomorph mapping Θ: $E(G) \to E(H)$ such that $\forall e \in E(G)$, $\forall e' \in E(H)$: $\Phi(e) = e' \Leftrightarrow \Theta(e) = e'$. Let an arbitrary edge-labeling of G be Λ. We show that there is a labeling Λ' of H corresponding to Λ such that $\Psi_C(\Lambda) = \Psi_C(\Lambda')$. Suppose the edge-labeling Λ' of H such as $\Theta = \Theta\Lambda^{-1}\Lambda = \Lambda'^{-1}\Lambda = \Lambda'^{-1}I_m\Lambda$ and $\Lambda' = \Lambda\Theta^{-1}$. Let the Ψ-formulas of Λ and Λ' be Ψ and Ψ' respectively. For all number sequences $x \in \Sigma_3^m$, the edge-isomorph mapping Θ maps x in Ψ to the same x in Ψ' through the identity I_m. Consequently every subgraph $G_\Lambda(x)$ in Ψ is mapped to the corresponding subgraph $H_{\Lambda'}(x)$ in Ψ' by the mapping Θ. Hence for all $x \in \Sigma_3^m$, $\Psi_C(G_\Lambda(x)) = \Psi_C(H_{\Lambda'}(x))$. This yields $\Psi_C(\Lambda) = \Psi_C(\Lambda')$. ■

6.4 *Two connected graphs G and H are isomorphic iff* $Y_C(G)$ *equals* $Y_C(H)$.

Proof: Since $\Psi_C(G)$ and $\Psi_C(H)$ are both edge-connected Ψ numbers, if $\Psi_C(G) = \Psi_C(H)$, then by **6.2**, G and H are isomorphic. The converse. Without loss of generality we assume that G and H are isomorphic but $\Psi_C(G) > \Psi_C(H)$. Then by **6.3** there must be an edge-labeling Λ' of H such that $\Psi_C(G) = \Psi_C(\Lambda') > \Psi_C(H)$. This conflicts the hypothesis that $\Psi_C(H)$ is the maximum edge-connected Ψ number of H. Hence the statement is deduced to be true. Note that this theorem declares that $\Psi_C(G)$ is a complete invariant of graphs. ■

6.5 Edge-Version Kelly-Ulam Conjecture: *Two connected graphs G and H of size m ³ 4 are isomorphic iff there exists a mapping Q: E(G) ® E(H) such that for all edges e Î E(G), G - e is isomorphic to H - Q(e).*

Failed Proof: Assume G and H are isomorphic. Then by **6.2**, there exist edge-labelings Λ of G and Λ' of H such that $\Psi_C(\Lambda) = \Psi_C(\Lambda')$, and by the definition of edge-connected Ψ numbers, ${}^\forall i \in \sigma_m$, $\Psi_C(G_\Lambda(i)) = \Psi_C(H_{\Lambda'}(i))$. Hence by **6.2** ${}^\forall i \in \sigma_m$, $G_\Lambda(i) \cong H_{\Lambda'}(i)$. Let e_i and e'_i be the edges such as $\Lambda(e_i) = \Lambda'(e'_i) = i$. Then $\Theta(e_i) = e'_i$, and ${}^\forall i \in \sigma_m$, $G - e_i \cong H - \Theta(e_i)$, where Θ is a mapping from $E(G)$ onto $E(H)$ such as $\Theta = \Lambda'^{-1}\Lambda$. Next we prove the converse, i.e., if G and H are non-isomorphic, then for any mapping Θ: $E(G) \to E(H)$, there exists an edge e_i in $E(G)$ such that $G - e_i$ and $H - \Theta(e_i)$ are non-isomorphic. Assume $G \not\cong H$. Then by **6.4**, $\Psi_C(G) \neq \Psi_C(H)$ and by **6.3** and **6.2**, and from the definition of edge-connected Ψ numbers, ${}^\forall\Lambda$, ${}^\forall\Lambda'$, ${}^\exists i \in \sigma_m$, $\Psi_C(G_\Lambda(i)) \neq \Psi_C(H_{\Lambda'}(i))$. Hence

$$ {}^\forall\Theta = \Lambda'^{-1}\Lambda,\ {}^\exists i \in \sigma_m,\ {}^\forall\Lambda \in \Lambda^G,\ {}^\forall\Lambda' \in \Lambda^H,\ \Psi_C(G_\Lambda - \Lambda^{-1}(i)) \neq \Psi_C(G_\Lambda - \Theta(\Lambda^{-1}(i))), $$

where Θ is a mapping: $E(G) \to E(H)$. Consequently ${}^\forall\Theta$, ${}^\exists e_i$ Î $E(G)$, $\Psi_C(G - e_i) \neq \Psi_C(H - \Theta(e_i))$. Then by **6.4**, ${}^\forall\Theta$, ${}^\exists e_i$ Î $E(G)$, $G - e_i \not\cong H - \Theta(e_i)$. ■

Remark: We encountered the dead end again. The reasoning above is wrong as was in the other proofs previously mentioned. The order of the quantifiers ${}^\forall\Lambda$, ${}^\forall\Lambda'$, ${}^\exists i$ cannot be rearranged like ${}^\exists i$, ${}^\forall\Lambda$, ${}^\forall\Lambda'$. The situation has been unchanged at all even though we came into the edge-reconstruction. In a sense it is much more critical than the vertex-reconstruction. It is that all the edge-deleted subgraphs are isomorphic and all the deleted edges are necessarily contradicting. Suppose two isomorphic graphs G and H, and an edge-isomorph mapping Θ: $E(G) \to E(H)$. We say edges $e \in E(G)$ and $e' \in E(H)$ are identical if $\Theta(e) = e'$. On the other hand if G and H are not isomorphic but $G - e$ and $H - e'$ are isomorphic, then we say that edges e and e' are pseudoidentical. Similarly identical vertices and pseudoidentical vertices are defined. It can be said that with respect to two graphs G and H of a counterexample of the edge-version K-U, every edges in the graphs are pairwise pseudoidentical.

7. Edge-Version Kelly-Ulam Theorem

Consider the complement of a counterexample of edge-version K-U. If graphs G and H is a counterexample pair of edge-K-U, then it is of course that $\overline{G}$ and $\overline{H}$ is also a counterexample pair. Because if the subgraphs of G and H are pairwise isomorphic, then of course the complement of them are isomorphic, while $G \not\cong H$ implies $\overline{G} \not\cong \overline{H}$. Hence $\overline{G}$ and $\overline{H}$ is also a counterexample pair. However a curious thing happens here such that the sets of subgraphs of $\overline{G}$ and $\overline{H}$ are not the edge-deleted subgraphs but so to say the edge-augmented subgraphs. Now let us call such a counterexample with opposite direction an inverse-phase counterexample. On the contrary an ordinary counterexample shall be called a normal-phase counterexample. Graphs are classified into 4 subclasses as follows.

(1) Reconstructible and edge-reconstructible graphs
(2) Non-reconstructible but edge-reconstructible graphs
(3) Normal-phase edge-non-reconstructible graphs
(4) Inverse-phase edge-non-reconstructible graphs

A few questions arise: Are there any edge-counterexamples such as of both normal-phase and inverse-phase? Are there normal-phase counterexamples with edges more than $n(n - 1) / 4$ or any inverse-phase counterexamples with edges less than $n(n - 1) / 4$? Fortunately L. Lovász already answered this question showing that if a graph has more edges than its complement then it is edge-reconstructible [26]. In other words, there exists neither normal-phase counterexamples of size $m > n(n - 1) / 4$ nor inverse-phase counterexamples of size $m < n(n - 1) / 4$. So queer and bizarre situation it is, and enough to give rise to a doubt for the existence of a counterexample of the edge-version K-U.

We should notice that the K-U problem has two faces, one is the K-U and the other is the Reconstruction. The first face regards two non-isomorphic graphs G and H, and the second is for an independent graph G itself. The relation between these two phases is not necessarily self-evident. Only it can be said to reflect the problem to distinguish between oneself and others. We will dig up the problem a little further. Now we introduce three more Ψ numbers, edge-labeled Ψ numbers Ψ_E, edge-deck Ψ numbers Ψ_D, and edge-fragment-deck Ψ numbers Ψ_F. The first Ψ numbers is defined for edge-labeled graphs and the rest are for unlabeled graphs.

$$\Psi_E(\Lambda) := \prod_{i=1}^{r} P(i)^{\Psi_C(G_\Lambda[i])}, \; i < j \Leftrightarrow \Psi_C(G_\Lambda[i]) \le \Psi_C(G_\Lambda[j]), \text{ where}$$

r is the number of the connected components of G_Λ,
$G_\Lambda[i]$ is the i-th connected component of the edge-labeled graph G_Λ, and
$\Lambda \in \Lambda^G$, G_Λ is an edge-labeled graph of G labeled by Λ.

$$\Psi_E(G) := \Psi_E(\Lambda_0) \ge \Psi_E(\Lambda) : {}^\forall\Lambda \in \Lambda^G, \text{ where } G \text{ is an unlabeled graph.}$$

The edge-labeled Ψ number $\Psi_E(\Lambda)$ can be regarded as an ordered sequence of edge-connected Ψ numbers Ψ_C of the connected components of G_Λ, where $G_\Lambda[i]$ denotes a connected component of the edge-labeled graph G_Λ and the Ψ numbers $\Psi_C(G_\Lambda[i])$ are arranged in the order of the value in the Ψ-formula $\Psi_E(\Lambda)$. Each edge-labeling Λ of G has its Ψ number $\Psi_E(\Lambda)$ and $\Psi_E(G)$ is the maximum value among them.

$$\Psi_D(pK_1) := 2^p. \; \Psi_D(G) := \prod_{i=1}^{r} P(i)^{\Psi_E(G-e_i)}, \; r = |E| > 0, \text{ and } i < j \Leftrightarrow \Psi_E(G - e_i) \le \Psi_E(G - e_j).$$

An edge-deck Ψ number is a collection of the edge-labeled Ψ numbers Ψ_E of the edge-deleted subgraphs of G. $\Psi_D(G)$ represents the condition of the edge-version K-U straightway. It is easy to see that with respect to two graphs G and H, if $\Psi_D(G)$ and $\Psi_D(H)$ are coincident, then all of edge-deleted subgraphs of G and H are pairwise isomorphic.

$\Psi_F(pK_1) := 2^p$, $\Psi_F(pK_1+K_2) := 2^p\times 3$, $\Psi_F(pK_1+2K_2) := 2^p\times 5$, $\Psi_F(pK_1+P_2) := 2^p\times 7$, $\Psi_F(pK_1+3K_2) := 2^p\times 11$, $\Psi_F(pK_1+K_2+P_2) := 2^p\times 13$, $\Psi_F(pK_1+P_3) := 2^p\times 17$, $\Psi_F(pK_1+K_{1,3}) := 2^p\times 19$, and $\Psi_F(pK_1+K_3) := 2^p\times 23$,
where p is the number of the isolated vertices in the graphs.

$$\Psi_F(G) := \prod_{i=1}^{r} P(i)^{\Psi_F(G-e_i)}, \; r = |E| \ge 4, \text{ and } i < j \Leftrightarrow \Psi_F(G - e_i) \le \Psi_F(G - e_j).$$

The edge-fragment-deck Ψ numbers Ψ_F is defined recursively for all k-edge-deleted subgraphs of G with $k \le m - 3$. Each term $\Psi_F(G - e_i)$ in the formula $\Psi_F(G)$ may not represent the edge-deleted subgraph $G - e_i$ itself, but rather represents the deck of the subgraph. Even if $\Psi_F(G)$ and $\Psi_F(H)$ are coincident, it is not always that all of edge-deleted subgraphs of G and H are pairwise isomorphic. It just says that the sub-decks represented by the Ψ numbers $\Psi_F(G - e_i)$ and $\Psi_F(H - e'_i)$ are pairwise coincident. In the first formulas above, p is the number of the isolated vertices in the graph. All of k-edge-deleted subgraphs appeared in the Ψ-formulas of a graph G of order n have the same number n of vertices. Obviously any graph G has only one $\Psi_D(G)$ and only one $\Psi_F(G)$.

Let $K(n)$ denote the set of all graphs of order n and $K(n,m)$ denote the set of all graphs of order n and size m, where $K(n,m) \subseteq K(n)$. We may call the graph set $K(n)$ the universal graph set (with order n). Let an arbitrary Ψ_F number be ψ, then $K_F(\psi)$ denotes the set of graphs whose Ψ_F numbers are ψ. With respect to the graph set $K_F(\psi)$ of an edge-fragment-deck Ψ number ψ, if $|K_F(\psi)| = 1$, then we say that $K_F(\psi)$ is a proper-graphs and ψ is a proper Ψ_F number. In the case of $|K_F(\psi)| > 1$, we say that $K_F(\psi)$ is a contradict-graphs and ψ is a contradict Ψ_F number. As is easily to be seen, an edge-fragment-deck Ψ number ψ can be regarded as a set of all of Ψ_F numbers which appeared in the Ψ-tree of the Ψ_F number ψ. Suppose two distinct Ψ_F numbers ψ_1 and ψ_2. If the Ψ_F number ψ_2 appears in the Ψ-tree of ψ_1, then we say that ψ_1 contains ψ_2 and write $\psi_2 \subset \psi_1$.[11]

7.1 *Two graphs G and H are isomorphic iff* $Y_E(G)$ *equals* $Y_E(H)$.

Proof: Without loss of generality we assume that the number of the connected components of G and H are same. Let k be the number of the connected components of G and H, and $G[i]$ and $H[i]$ be the connected components of them respectively. Assume G and H are isomorphic. Then there exists a mapping $\Phi: \sigma_k \to \sigma_k$ such as ${}^\forall i$: $G[i] \cong H[\Phi(i)]$. Hence by **6.4**, ${}^\forall i$: $\Psi_C(G[i]) = \Psi_C(H[\Phi(i)])$. This yields $\Psi_E(G) = \Psi_E(H)$. Since the components $G[i]$ and $H[i]$ are not connected this equation is valid regardless their labelings. Assume $\Psi_E(G) = \Psi_E(H)$. Then by the definition of edge-labeled Ψ numbers Ψ_E, ${}^\forall i$: $\Psi_C(G[i]) = \Psi_C(H[i])$, and by **6.4**, ${}^\forall i$: $G[i] \cong H[i]$. This yields $G \cong H$. Note that this theorem declares that $\Psi_E(G)$ is a complete invariant of graphs. ■

[11] We consider $\Psi_F(G - e_i) \subset \Psi_F(G)$ and regard that the $\subset$ relation is transitive. Then we say that $\Psi_F(G)$ is a set of Ψ_F numbers and contains all of Ψ_F numbers appeared in the Ψ-tree of $\Psi_F(G)$. Accurately $\Psi_F(G)$ is said to be a set of Ψ_F numbers of small graphs of size 3, but $\subset$ is not necessarily equivalent to the inclusion sign $\subseteq$ in the Set Theory. Shall we use the term category here?

7.2 *For any two graphs G and H, the following two statements are equivalent.*

(1) *G and H are isomorphic iff* $\Psi_D(G)$ *equals* $\Psi_D(H)$.
(2) *G and H are isomorphic iff* $\Psi_F(G)$ *equals* $\Psi_F(H)$.

Proof: Without loss of generality we assume that the size of graphs is larger than 3. From the definition of the edge-deck Ψ numbers and the edge-fragment-deck Ψ numbers, it is easy to see that if G and H are isomorphic, then $\Psi_D(G) = \Psi_D(H)$ as well as $\Psi_F(G) = \Psi_F(H)$. Further it is sure that if $\Psi_D(G) = \Psi_D(H)$, then $\Psi_F(G) = \Psi_F(H)$. Because by the definition of edge-deck Ψ numbers, whenever $\Psi_D(G) = \Psi_D(H)$, it comes to be $^\forall i \in \sigma_m$: $\Psi_E(G - e_i) = \Psi_E(H - e'_i)$. Consequently by **7.1**, $^\forall i \in \sigma_m$: $G - e_i \cong H - e'_I$, hence $^\forall i \in \sigma_m$: $\Psi_F(G - e_i) = \Psi_F(H - e'_I)$. This yields $\Psi_F(G) = \Psi_F(H)$. Thus (2) $\Rightarrow$ (1) is apparent. Next for the direction (1) $\Rightarrow$ (2), assume (1) is true. We will prove (2) by a mathematical induction.

Assume that there exists a natural number k, and (2) is true for any graphs G and H of size $\leq k$. Suppose two graphs G and H of size $k + 1$, and assume $\Psi_F(G) = \Psi_F(H)$, then by the definition of edge-fragment-deck Ψ numbers, $^\forall i \in \sigma_m$: $\Psi_F(G - e_i) = \Psi_F(H - e'_I)$. Since the size of $G - e_i$ and $H - e'_i$ is k, by the induction hypothesis, $G - e_i \cong H - e'_i$. Hence by **7.1**, $\Psi_E(G - e_i) = \Psi_E(H - e'_i)$. This yields $\Psi_D(G) = \Psi_D(H)$ and by the assertion (1), G and H are isomorphic. It is easily certified that (2) is true for small graphs of size ≤ 4 and it completes the induction. Thus we have (1) $\Rightarrow$ (2). ■

7.3 *Let the edge-fragment-deck* Ψ *number of a complete graph* K_n *be* ψ_n*. then for any natural number n, the graph set* $K_F(\psi_n)$ *is a proper-graphs.*

Proof: Assume $K_F(\psi_n)$ is not a proper graphs. Then there must be at least two graphs which are not isomorphic in the $K_F(\psi_n)$. However there exists no other graphs than K_n in the graph class $K(n, n(n - 1) / 2)$ and obviously $K_F(\psi_n) \subseteq K(n, n(n - 1) / 2)$. Therefore the graph set $K_F(\psi_n)$ must be a singleton set, and it comes to be a proper graphs. ■

7.4 *Suppose two edge-fragment-deck* Ψ *numbers* ψ_1 *and* ψ_2*. If* ψ_1 *contains* ψ_2 *and* $K_F(\psi_2)$ *is a contradict-graphs, then* $K_F(\psi_1)$ *is also a contradict-graphs.*

Proof: Assume $K_F(\psi_1)$ is a proper graphs, i.e., every graphs in the graph set $K_F(\psi_1)$ are isomorphic. Since $K_F(\psi_2)$ is a contradict-graphs, $K_F(\psi_2)$ contains at least two non-isomorphic graphs G_2 and H_2. Let two graphs contained in $K_F(\psi_1)$ be G_1 and H_1 such that G_1 has a G_2 as its subgraph and respectively H_1 has a H_2 as its subgraph. We assume that even if G_1 and H_1 have a plural of G_2 and H_2, G_1 never has a H_2 and H_1 never has a G_2. Suppose edge-labeled Ψ numbers $\Psi_E(G_1)$ and $\Psi_E(H_1)$ and let Ψ and Ψ' be the Ψ-trees of $\Psi_E(G_1)$ and $\Psi_E(H_1)$ respectively.

Assume G_1 and H_1 are connected. Then all of edge-induced subgraphs of the graphs G_1 and H_1 of size ≥ 3 appear in the Ψ-trees. To be $\Psi_E(G_1)$ and $\Psi_E(H_1)$ coincident, every corresponding nodes have an exactly same value of edge-connected Ψ numbers Ψ_C. Since G_1 and H_1 have a pair of non-isomorphic subgraphs G_2 and H_2, it is sure that $\Psi_E(G_1)$ and $\Psi_E(H_1)$ cannot be equal. Assume G_2 and H_2 are connected. Then by **6.4**, $\Psi_C(G_2) \neq \Psi_C(H_2)$. Of course there exist no labelings which make them equal. Now assume G_2 and H_2 are disconnected, then there always exist some non-isomorphic connected components of G_2 and H_2, and they cannot have the same edge-connected Ψ numbers.

This reasoning is valid even if G_1 and H_1 are disconnected. Hence by **7.1**, G_1 and H_1 are not isomorphic. This contradicts the hypothesis that $K_F(\psi_1)$ is a proper graphs. To confirm this reasoning, consider the case when all of subgraphs of G_1 and H_1 are pairwise isomorphic. Even in such a case, we can replace an occurrence of G_2 (H_2) in H_1 by H_2 (respectively G_2) and get a non-isomorphic pair of graphs G_1 and H_1. This operation does not change the edge-fragment-deck Ψ number $\Psi_F(H_1)$ but necessarily changes the edge-labeled Ψ number $\Psi_E(H_1)$. Consequently the graph set $K_F(\psi_1)$ of a Ψ_F number ψ_1 which contains a contradict Ψ_F number is also a contradict-graphs. ■

7.5 *Two graphs G and H are isomorphic iff* $\Psi_F(G)$ *equals* $\Psi_F(H)$.

Proof: Without loss of generality we assume that the size of graphs is larger than 3. From the definition of edge-fragment-deck Ψ numbers, it is easy to see that if G and H are isomorphic, then $\Psi_F(G) = \Psi_F(H)$. Then we prove the converse by a reductio method, i.e., we prove that there does not exist a counterexample of the statement saying that if G and H are non-isomorphic, then $\Psi_F(G) \neq \Psi_F(H)$. Assume that there exists a pair of counterexample graphs G and H of size ≥ 4 such as $G \not\cong$ H and $\Psi_F(G) = \Psi_F(H) = \psi$. Let the order and the size of G and H be n and m respectively, then

$G,H \in K_F(\psi) \subseteq K(n,m) \subseteq K(n)$. Suppose the complete graph K_n of order n and let the Ψ_F number of K_n be ψ_n. By the definition of the edge-fragment-deck Ψ numbers, it is obvious that ψ_n contains every Ψ_F numbers of any graphs of size ≥ 4 in the universal graph set $K(n)$, i.e., $^\forall G \in K(n), |E(G)| \geq 4: \Psi_F(G) \subset \psi_n = \Psi_F(K_n)$. Consequently it turns out to $\psi \subset \psi_n$. By the hypothesis, $K_F(\psi)$ is a contradict-graphs, then by **7.4**, ψ_n must be a contradict-graphs. However from **7.3**, a complete graph cannot be in a contradict-graphs. Hence a contradiction, and the consequence comes to be that the statement is true. Note that by this theorem and **7.2**, $\Psi_F(G)$ and $\Psi_D(G)$ are complete invariants of graphs. ■

7.6 Edge-Version Kelly-Ulam Theorem: *Two graphs G and H of size m ³ 4 are isomorphic iff for each i £ m, $G - e_i$ is isomorphic to $H - e'_i$, where e_i and e'_i are edges of G and H respectively.*

Proof: By **7.2** and **7.5**, the statement (1) "G and H are isomorphic iff Ψ_D(G) equals $\Psi_D(H)$" is true. Consequently if $G \cong H$, then $^\forall i \in \sigma_m$: $\Psi_E(G - e_i) = \Psi_E(H - e'_i)$, and by **7.1**, $^\forall i \in \sigma_m$: $G - e_i \cong H - e'_i$. Next for the converse, assume $^\forall i \in \sigma_m$: $G - e_i \cong H - e'_i$, then $^\forall i \in \sigma_m$: $\Psi_E(G - e_i) = \Psi_E(H - e'_i)$. This yields Ψ_D(G) $= \Psi_D(H)$. Hence by (1), we get $G \cong H$. ■

8. Conclusion

We finally succeeded to prove the edge-version K-U using four kinds of Ψ numbers, edge-connected Ψ numbers Ψ_C, edge-labeled Ψ numbers Ψ_E, edge-deck Ψ numbers Ψ_D, and edge-fragment-deck Ψ numbers Ψ_F. Lemma **6.1** is the cornerstone of the formulation of all the edge Ψ numbers. It was exhibited that the lemma **6.1** is equivalent to the Whitney's Theorem. The Ψ_E is the most standard edge Ψ numbers and it is assured that if $\Psi_E(G) = \Psi_E(H)$, then G and H are always isomorphic. An edge-deck Ψ number $\Psi_D(G)$ can be regarded as a collection of edge-labeled Ψ numbers $\Psi_E(G - i)$ and it represents the edge-deck of G. If $\Psi_D(G) = \Psi_D(H)$, then all of edge-deleted-subgraphs are isomorphic. However if edge-K-U is invalid, it is possible that G and H are non-isomorphic. The edge-fragment-deck Ψ numbers Ψ_F is defined recursively and represents so to say the fragmented deck of G. The equivalence of the two edge Ψ numbers, Ψ_D and Ψ_F was proven.

We call a Ψ_F counterexample a contradict Ψ_F number. For the edge-fragment-deck Ψ numbers, it is possible that not only graphs G and H are non-isomorphic, but also there exist subgraphs which are not isomorphic, even if their edge-fragment-deck Ψ numbers are entirely equal. Because it is possible that a contradict Ψ_F number contains contradict Ψ_F numbers. Two graphs with the same Ψ_F number possibly have non-isomorphic subgraphs since a contradict Ψ_F number indicates a set of non-isomorphic graphs. This is quite different from edge-deck Ψ numbers Ψ_D. A counterexample Ψ_D never contains a non-isomorphic subgraph. We proved that there exist no contradict Ψ_F numbers by showing that the assumption of the existence of contradict Ψ_F numbers results an absurdity.

What about the vertex-version K-U. It can be said that we can define vertex-deck Ψ numbers and vertex-fragment-deck Ψ numbers as well as in the edge-version K-U. Furthermore we can prove similar theorems to **7.2** and **7.4** for it. However we don't know how to get the theorem **7.5** as we have not the alternative lemma **7.3** for the vertex-version K-U. The vertex-fragment-deck Ψ number of any complete graph K_n does not contain a contradict Ψ number. Conversely it is possible that there exist an infinite number of vertex-fragment-deck Ψ numbers containing a contradict Ψ number. In consequence, we cannot help considering that it is probable that a K-U counterexample exists. Although it seems very difficult to compose a counterexample.

We separate a labeled graph G into two parts, F_0 and F_1 such as a vertex-deleted subgraph and the rest of the graph. The F1-friction is a contradiction at the F_1 part which represents the adjacency between the deleted vertex and other vertices. This part simply forms a star.[12] Of course for any pairwise isomorphic subgraphs $G - i$ and $H - i$, the two stars are always isomorphic as the numbers of edges are same, but some of the edges in the stars are at bad positions on the labelings. A remarkable point is that each vertex-deleted subgraph pair of a K-U counterexample must have the F1-friction with no exception. It is really difficult to make the requirement compatible with the ultimate condition such as the graphs are nearly isomorphic but just separated by skin-deep. K-U counterexamples are isolated each other and floating like a big soap-bubble in the sky. How can we find such a counterexample? We dare to say that the K-U Conjecture is not merely a hard problem but should be regarded as a paradox. We left our failed proofs as it was in the article for the sake of exposing the dead end of the proofs.

[12] A $K_{1,n}$ is called a star.

Incidentally the relation between pseudoidentity and pseudosimilarity is not so clear, where two vertices u and v are pseudoidentical if G and H are not isomorphic but $G-u$ and $H-v$ are isomorphic. We know that a graph G cannot have all its vertices mutually pseudosimilar. On the other hand the pseudoidentity in a counterexample occurs at every vertices in the graph. But it is also known that there exist graphs in which every vertex has a pseudosimilar mate [20]. Is F1-friction equivalent to pseudosimilarity? At first sight it seems not likely. Pseudosimilarity is defined with respect to graph automorphism on the graph itself, while pseudoidentity regards non-isomorphism between distinct graphs. Nevertheless it may happen that they are involving each other since the reconstruction is a problem on a graph G itself. The pseudosimilarity still has a great chance to refute the existence of F1-friction.[13]

The difficulty of the K-U Conjecture looks so supernatural and desperate. Like a tall and cruel stone wall, it blockades human, our mortal existence far from the solution. What shall we name this barrier? A book written by Ilya Prigogine a Nobel laureate chemist, and Isabelle Stengers a historian of science says, "We cannot provide a situation evolving to the past because infinite information is indispensable to reverse the direction of the time.". They call it "entropy barrier" comparing the "velocity of light" as the upper bound of the speed of signal transmission which is the basis of Einstein's relativity and prohibits a time-travel to outstrip the light.[14] In this sense, to find a counterexample of K-U may be comparable to Michelson-Morley experiment of measuring the velocity of light which ultimately denied the concept of ether. They ask "What's the particular structure of a dynamic system which are able to distinguish the past and the future? How much the degree of the minimum complexity to be necessary for that?". If we succeed to discover a counterexample of K-U, it must be a very primitive answer to this question from mathematics. However it is somehow unbelievable that it will be given as some constant number.[15] So our withheld estimation is infinite... Our final conclusion comes to be that K-U is a paradox deeply relevant to the irreversibility of the time.

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[13] Conversely it can be considered that there exists a counterexample which has no pseudosimilar vertices.

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[15] Order of graphs may not be suitable to measure K-U complexity. Perhaps a new measurement will become necessary.

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