Strongly Intransitive Graphs and The Perfect Graph Conjecture

Dedicated to
TIBOR GALLAI
the most outstanding pioneer
on transitivity concept in graph theory

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Abstract

An orientation of an undirected graph G is a directed graph D obtained by giving an orientation to every edges of G. An orientation D is complete iff whenever there are edges ab, bc, there is an edge ac in D. A graph G is transitive iff G has a complete orientation. If there are edges ab, bc and an anti-path (a,c) (i.e., a path in the complement of G) in the subgraph induced by the neighbors of b in G, the triplet (a,b,c) is called a pivot on G and the middle vertex b is the pivot-vertex of it. We proved in **T14** that a graph G is transitive iff G has no pivot-cycles (i.e., a closed odd sequence of pivot-vertices) and got a polynomial-time algorithm to recognize a transitive graph and to construct a complete orientation of the graph.

If a graph G has a pivot-cycle and the size of the cycle equals the size of G, we say that the pivot-cycle is spanning. An intransitive graph G is called minimally intransitive iff G becomes transitive by removing any vertex of G. We say a graph G is strongly intransitive iff both G and the complement of G are minimally intransitive and have a spanning pivot-cycle. We proved in **T24** that odd holes and odd antiholes are strongly intransitive graphs. Regarding the long standing conjecture presented by Berge known as the Strong Perfect Graph Conjecture, we propose the following:

C1: *A strongly intransitive graph is minimally imperfect*.
C2: *A strongly intransitive graph is either an odd hole or an odd antihole*.
C3: *A graph is perfect iff it has no strongly intransitive graphs*.

Comments

This paper is still in the state of draft as easily to be seen. Currently we have seven pending theorems,**T15**, **T16**, **T26**, **T28**, **T30**, **T31**, **T170** relevant to our conjectures. We performed this study on the net, have found so many colleagues and teachers there and get a plenty amount of precious information from enormous on-line stuffs and books. It happened at a mailing list theory-edge* in the end of the 20th century and has been continuing up to now.

On Octover 20, 2000, a mail titled "*odd holes, perfect graphs, and theta fn*" was posted to the list. It was written by E. Lehman, a member of the list to digest some indigestion in the process of reviewing A. Plotnikov's paper [19]. The paper regarded the *Minimum Clique Partition Problem* and presented an approach through transitivity and some hereditary property of graphs.

Due to the mail we came to know those things for the first time such as a *hole*, *perfect graphs*, *the theta function*, and *Berge's conjecture*. The theta function, sometimes called *the sandwich function*, calculates Lovász's number θ between the largest clique size ω and the chromatic number χ of a graph in polynomial time [13]. It is known *NP-complete* in general to calculate those numbers. However as ω and χ coincide for a perfect graph (just the definition of perfect graphs), one can get the solution in polynomial time. This was presented by Grötschel, Lovátz, and Schrijver in 1981 [7]. Then we understood that *it* would be solved anyway.

But what's an odd hole? It was the question. After a mean while, we reached at the notion *OZ-cycle*. A transitive contradiction caused by an OZ-cycle in a *plotnikov digraph*, i.e., an *orientation* of an undirected graph, is inevitable by any means, surely a very strange cycle. We ascertained that *odd holes* and *odd antiholes* are the representatives of this kind (**T18**). But how can we formulate and give it a reasoning?

Finally we found out a simple fact that the transitive relation is entirely *local*. We know that an eternal triangle is fatal at any time. We summarized this insight to **T11**: the Triple Contradiction Theorem which states "*a directed graph D is transitively complete iff every vertex triple of D is not in triple contradiction*". Some authors regard it as *a well-known obvious observation*. Well, how can the locality be transported to the remote?

Perhaps the answer is "*by paths*". We found two kinds of particular paths, *zigzag paths* and *twisted paths*. Those are strands of twisted triple strings. Roughly speaking, all of paths in the *pivot-map*** of the *delta graph*** of *G* are zigzag and all of paths in the *core-map*** are twisted (**T152)**. If the all paths are linear, the transitive orientation is easily completed but it returns to itself in circulation. An *OZ-cycl*e, i.e., odd zigzag cycle returns the transitive contradiction.

The OZ-cycle was found first by Ghouillà-Houri, 1962 [4], and independently by Gilmore and Hoffman, 1964 [5]. Our earliest conclusion was that *a graph G is transitive iff G has no OZ-cycles*. This assertion holds (**T25**). However it was disproved temporarily by G. Stertenbrink. (At the time we cannot help considering that a cycle must be *elementary*. Otherwise we had already solved it then.) He showed a counter example which has no elementary OZ-cycles but is intransitive. It was the blessed complement of C_6. Soon the fact was uncovered that *every antihole greater than four is intransitive* (**T20**).

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* theory-edge: http://groups.yahoo.com/group/theory-edge

** pivot-part, core-part, delta graph: Shall be described later.

To prove it, we invented *Δ-cycle* and *round-fan,* and succeeded to show that all antiholes except $\sim C_4$ (i.e. the complement of a cycle of size 4) are round-fans in fan-contradiction. Consequently the revised **T14** came to be "*a graph G is transitive iff it has no OZ-cycle and no round-fan in fan-contradiction*". This formulation looked like too artificial and this time a counterexample was found by the author himself. Gradually we noticed that to formulate transitivity, we have to deal with the complements of graphs. *A fan* is a subgraph of G induced by $V_f \cup \{p\}$, where V_f is the vertex set of a connected component of the induced subgraph of $\sim G$ by the neighbors of the vertex p in G. The notion of *fan* was discovered by Tibor Gallai in 1963 [3], although Gallai himself did not give it a name.

Now, we have the final version of **T14**, i.e., "*a graph G is transitive iff G has no pivot-cycles*". Pivot-cycle is a generalization of OZ-cycle and a *pivot* is a kind of fan. So we and Gallai share the notion *pivot-cycle*. Some literature deal with *asteroids* with respect to anti-transitive graphs (i.e., co-comparability graphs). We know that an asteroid is nothing but a reversed pivot-cycle. Gallai [3] presented 19 patterns of *minimally intransitive graphs*. Among them, 1 is odd hole, 3 are odd cycles with two neighbors and remaining 15 are (the complement of) 3-asteroids. He proved every intransitive graph contains one of these 19 minimally intransitive graphs. (See Appendix, the Gallai's Gamma Table.)

We construct the *delta graph* G_d of G. A *delta* is a *2-path in G* and a vertex of G_d is a delta of G. We call a vertex of G_d a *point* and an edge a *line*. If the edges in G of a delta pair forms a *3-path* in G, the corresponding point pair is joined in G_d. A delta (a,b,c) is called a *pivot* if there is an *anti-path* connecting a with c in the subgraph induced by the neighbors of the middle vertex b in G. A point in G_d is called a *pivot-point* if the corresponding delta is a pivot, otherwise called a *core-point*. Accordingly the delta graph G_d is partitioned into two parts of the *pivot-map* and the *core-map*. An odd cycle in the pivot-map corresponds to a pivot-cycle in G.

The following statements are equivalent for an undirected graph G (**T25**).
(1) *G is transitive.*
(2) *G has no OZ-cycles.*
(3) *G has no pivot-cycles.*
(4) *G has no elementary pivot-cycles.*
(5) *~G has no asteroids.*

The proof of (2) was presented by Ghouillà-Houri [4], Gilmore & Hoffman [5] in early 60'. (2) is the base of most succeeding studies including *implication class*[*] with respect to the transitivity. Gallai [3] solved (3), (4), (5). We proved them solely without using any preceding results. (In our formulation, the equivalence of (3) and (5) is given by the definition from the beginning. (4) remains for our homework.)

What the author is concerning is that none of literature mentioned the *loops* in OZ-cycles. In his humble opinion, to establish (2), it is necessary to allow loops in OZ-cycles. Otherwise infinite counterexamples will be inevitable. Adding to the above, Gallai [3] showed "G is transitive iff $\sim G$ has no *simple* asteroids". However the usage of "simple" in his paper is very particular and not usual. Some graphs G are minimally

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[*] implication class: Graph $G(V,E)$, binary relation Γ on E: $ab\Gamma a'b' \Leftrightarrow a = a'$ and $bb' \notin E$ or $b = b'$ and $aa \notin E$. The reflexive, transitive closure Γ^* of Γ is an equivalence relation on E, and partitions E into the implication classes of G [21].

intransitive but the asteroid in ~G is not *simple* in normal meaning.

Gallai [3] constructed a *fan graph* to prove (3) and (4). The f*an graph* F_g is a derived graph from G such that a vertex of F_g is a fan $f(p,aC)$, where p is a vertex of G and aC is the vertex set of a connected component of ~$N(p)$, and two vertices $f_1(p_1,aC_1)$ and $f_2(p_2,aC_2)$ of F_g are joined iff $p_1 \in aC_2$ and $p_2 \in aC_1$. An odd cycle in the fan graph F_g corresponds to a pivot-cycle of G.

Our proof of **T14** is done in a constructive way, i.e., by presenting algorithm A to construct a complete orientation of a transitive graph. The algorithm recognizes a transitive graph in polynomial time. The time complexity of algorithm A is estimated $O(n^6)$. The remarkable point of our algorithm is that once the delta graph is constructed, all of orientations is determined straightforwardly. The bottleneck of the algorithm is the initializing cost of the delta graph. Some linear time algorithms are already known for graph transitivity [17], [21]. They avoid such initial cost in a highly technical way and apply a divide & conquer method by decomposing the objective graph. We admit that the efficiency is not our principle goal.

Our main target is to prove positively *the Strong Perfect Graph Conjecture* proposed by Claude Berge [1], that is, "*a graph G is perfect iff neither G nor the complement contains an odd cycle of length at least five as an induced subgraph*. In spite of the dedicated enormous amount of studies, the conjecture is still open after 40 years. The course we chose to attack the *SPGC* peek is the *transitivity*. The author thinks the *perfection* of graphs and the *transitivity* have a very strong connection with each other. Some large subclasses of perfect graph class can be characterized by transitivity. Interval graphs is a subclass of anti-transitive graphs, permutation graphs is both transitive and anti-transitive graphs, and so on.

We provided three (incomplete) solutions, i.e., **Proof 1**, **2**, and **3** of **T17** for *SPGC*. **1** and **3** are based on the idea of getting a transitive / perfect supplement graph by adding edges to a graph which has no odd holes and no antiholes without increasing the maximum clique size. Wagler [22] showed in her Ph.D. thesis, there are such critically perfect graphs that cannot be reached by the deletion or the addition of one edge. This means that the Atlas of the perfect graphs is very intermingled like fractals.

The difficulty of course **3** is in the absence of an established method to recognize or compose a perfect graph. The course **1** is somehow hopeful as it substitutes the transitivity for the perfection. We start at the initial digraph D having all of vertices of G and no edges. Trivially the initial digraph D is transitively complete and we move an edge from G to D step by step. If D is not complete, add a supplemental edge to D until it becomes complete. If the addition of edges do not increase the maximum clique size till the end, we have done.

Currently we are concentrating to the course **2**, where we introduce the notion of *strongly intransitive*. An intransitive graph G is *minimally intransitive* iff G becomes transitive by removing any vertex of G. We say a graph G is *strongly intransitive iff both G and the complement of G are minimally intransitive and have a spanning pivot-cycle*. To solve the *SPGC*, we must prove the following 6 theorems.

T24: *Odd holes and odd antiholes are strongly intransitive graphs* .
T26✯ *A strongly intransitive graph is imperfect*.
T27: *A strongly intransitive graph is a minimally imperfect graph*.
T28✯ *The maximum clique size and the chromatic number of a graph which has no strongly intransitive graphs are coincident*.
T29: *A graph which has no strongly intransitive graphs is perfect*.

T30☆ *A strongly intransitive graph is either an odd hole or an odd antihole*.

Among them, **T24**, **T27**, and **T29** are already given proofs. **T26**, **T28**, and **T30** are unproved. Since the representation of strongly intransitive graphs is very clear, we suppose to solve **T26** and **T30** is very hopeful. **T28** may be not so easy. If we could prove **T30**, the probability for *SPGC* becomes very high, on the other hand, even if we proved **T26**, there remains some probability that the strongly intransitive graphs is not identified with the class of odd holes and odd anti-holes.

Berge [1] posed in 1960 one more conjecture called *the Weak Perfect Graph Conjecture*. It was proved by Lovátz, 1972 [16] and now known as *the Perfect Graph Theorem*. It states that *a graph is perfect iff its complement is perfect*. Between those two conjectures, Vasek Chvátal, 1984 [2] interposed a conjecture called *the Semi-Strong Perfect Conjecture*, which states "*if a graph has the P_4-structure of a perfect graph then it is perfect*". P_4 denotes a 3-path, i.e., an elementary path of length three. A graph G has the P_4-structure of a graph H if there is a bijection f between the set of vertices of G and the vertices of H such that a set S of four vertices in G induces the P_4 in G iff $f(S)$ induces a P_4 in H.

Since the complement of a 3-path is a 3-path again, the P_4-structure of a graph and its complement are isomorphic. Accordingly *SSPGC* implies *WPGC*. Chvátal [2] gave a rough proof for a theorem which states that *the only graphs having the P_4-structure of an odd cycle of length at least five are the cycle itself and its complement*, and showed *SPGC* implies *SSPGC* applying the theorem. This conjecture was proved by Reed [20] in 1987 and now called *the Semi-Strong Perfect Graph Theorem*.

Hougardy [11] showed that the Semi-Strong Perfect Theorem is rather weak for some graph classes to certify the perfection and asked "whether one can replace the P_4 in this theorem by some other graph". He answered to it by himself, "It is easily seen that the only possible candidates for such a result are the P_3 and its complement". We agree to this. Our *delta graph* is a kind of representation of P_3-structure.

Wing is the one more approach to use P_4-structures. An edge in a graph G is called a *wing* if it is one of the two non-incident edges of an induced P_4 in G. For a graph G: its *wing-graph* $W(G)$ is defined as the graph whose vertices are the wings of G and two vertices in $W(G)$ are connected if the corresponding wings in G belong to the same P_4. Hoàng [10] has conjectured that *a graph is perfect if its wing-graph is bipartite*. The graphs whose wing-graph is bipartite are called Hoang-graphs. Up to now his conjecture is still open.

Surely we have already a plenty of conjectures. I'm afraid that we are merely increasing the number of the unsolved. So we are going to solve our conjecture by ourselves, provided we can do it…Well, let us share the problem. The following is our proposal to the readers. Be enjoyed!

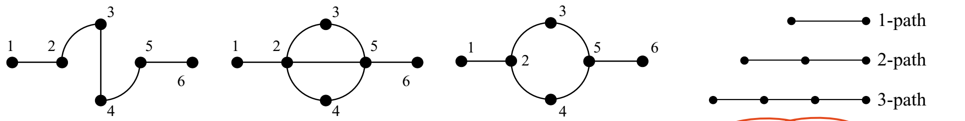
C1: *A strongly intransitive graph is minimally imperfect.*
C2: *A strongly intransitive graph is either an odd hole or an odd antihole.*
C3: *A graph is perfect iff it has no strongly intransitive graphs.*

M.N. March 17, 2001

Definition

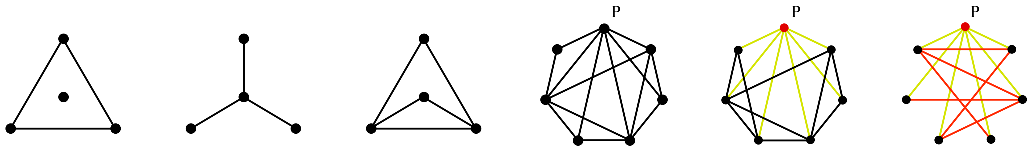
To solve the problem, we have to come out and go into three phases of graphs, undirected graphs, directed graphs and the complement of the graphs. To distinguish those objects, we provide the naming of ...*digraph* for directed graphs and *anti*... for the complements. Our definition for *paths* and *cycles* follows C.L. Liu [15]. Every subgraph in this article is a *vertex induced subgraph* unless mentioned explicitly. We say a graph *G has* a graph *H*. This implicitly represents that *H* is a *vertex induced subgraph* of *G*.

A ***path*** is an alternating sequence of vertices and edges incident with each other which begins and ends at vertices. We say a path is ***elementary*** if no vertex occurs more than once in the sequence. As well a path is ***simple*** if no edge occurs more than once in it. An ***anti-path*** in a graph *G* is a path in the complement of *G*. A ***k-path*** is an elementary path of length (i.e., the number of edges) *k*. **~*G*** denotes the complement of *G*.



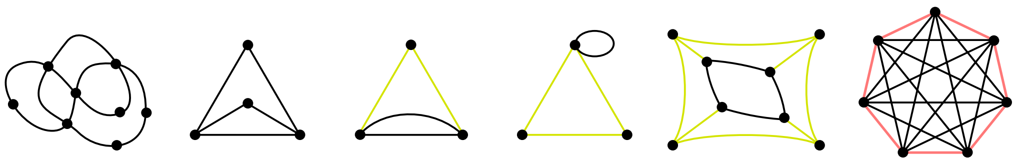
an elementary path (1,2,3,4,5,6) a simple path (1,2,3,5,4,2,5,6) a path (1,2,3,5,4,2,3,5,6)

A graph *G* is ***connected*** iff there is a path connecting a pair of vertices for all vertex pair in *G*. A graph *G* of size > 2 is called ***2-connected*** iff there are at least two distinct elementary paths connecting each pair of vertices. A graph *G* is ***anti-connected*** iff the complement of *G* is connected. ***N*(*v*)** denotes a subgraph of *G* induced by all the neighbors of the vertex *v*. Accordingly *~N(v)* is the subgraph of ~*G* induced by all the neighbors of *v* in *G*.

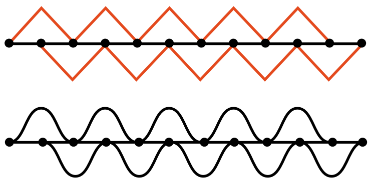


disconnected connected 2-connected G N(p) ~N(p)

A ***cycle*** is a closed path such that the initial vertex coincides with the end vertex. A ***multigraph*** is a graph allowed to have more than one edges joining the same two vertices. A ***simple graph*** has no such ***multi-edges***. A ***loop*** is an edge joining a vertex to itself. We assume that any vertex in either a simple graph or a multi-graph has no loops *except in the case when we consider OZ-cycles* in a simple graph. A vertex induced subgraph *H* of size > 3 of a graph *G* which is an elementary cycle having no chords is called a ***hole***. An ***antihole*** of *G* is a hole in the complement of *G*.

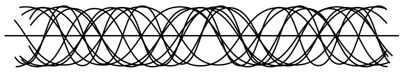

a simple cycle a simple graph multi-edges a loop a hole an antihole

An induced subgraph *Z* of a graph *G* is a ***zigzag path*** iff it has a spanning path $P_0(p_0,p_1,\ldots,p_n)$, and anti-paths $aP_1(p_0,p_2,\ldots)$, $aP_2(p_1,p_3,\ldots)$ such as the alternate vertex sequences of P_0, $V(P_0) = V(aP_1) \cup V(aP_2)$. An induced subgraph *T* of a graph *G* is a ***twisted path*** iff it has a spanning path $P_0(p_0,p_1,\ldots,p_n)$ and paths $P_1(p_0,p_2,\ldots)$, $P_2(p_1,p_3,\ldots)$ such as the alternate vertex sequences of P_0, $V(P_0) = V(P_1) \cup V(P_2)$. A twisted path *T* is ***strongly twisted*** iff every sub-path in *T* is twisted. Note: "***spanning*** *path*" not necessarily implies "elementary" here.



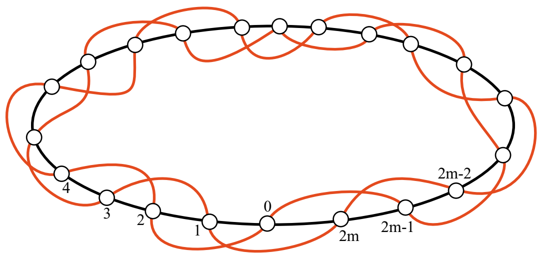
a zigzag path

a twisted path



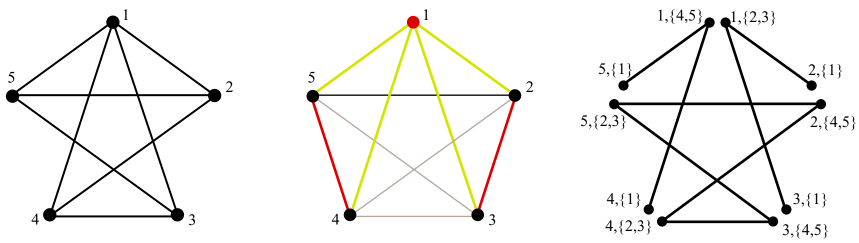
a strongly twisted path (image)

An induced subgraph Z of a graph G is an ***OZ-cycle*** $Z(Z_0,Z_1)$ if Z is a closed odd zigzag path in G, where Z has a spanning cycle $Z_0(p_0,p_1,\ldots,p_{2m})$, $m > 1$ and a spanning anti-cycle $Z_1(p_0,p_2,\ldots,p_{2m-1})$ of the alternate vertex sequence of Z_0. We call the spanning cycle Z_0 / Z_1 the ***front-cycle*** / ***rear-cycle*** of Z respectively. *The rear-cycle Z_1 is allowed to pass through loops*. We sometimes call the front-cycle Z_0 itself an OZ-cycle.



an OZ-cycle

A subgraph of a graph G induced by $V_f \cup \{p\}$ is a ***fan*** $F(p,V_f)$ on G, where p is a vertex of G, $V_f \subseteq V(N(p))$ and the subgraph induced by V_f is anti-connected in $N(p)$. The vertex p is called the ***fan's pivot***. A ***fan graph*** F_g is a derived graph from G such that a vertex of F_g is a fan $f(p,aC)$, where p is a vertex of G and aC is the vertex set of a connected component of $\sim N(p)$, and two vertices $f_1(p_1,aC_1)$ and $f_2(p_2,aC_2)$ of F_g are joined iff $p_1 \in aC_2$ and $p_2 \in aC_1$.



G

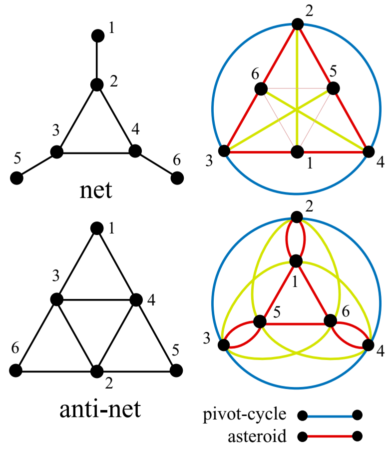
fans (1,{2,3}), (1,{4,5})

the fan graph of G

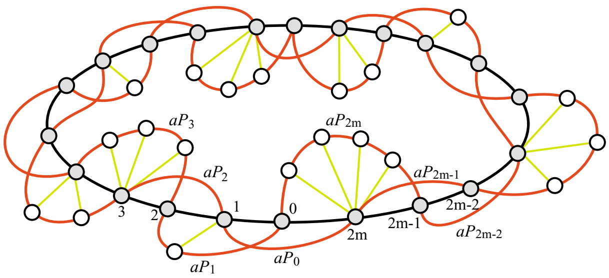
A ***delta*** in graph G is a *2-path* (a,b,c) in G. If an anti-path connects a with c in $N(b)$, the delta (a,b,c) is a fan $f(b,\{a,c\})$ and we call the fan f itself a ***pivot*** on G. The middle vertex b is called the ***pivot-vertex***. A path P of G is a ***pivot-path*** / ***core-path*** in G if every delta in P is a pivot / non-pivot respectively. Note: for a triangle (a,b,c): there are three distinct deltas. A delta (a,b,c) coincides with the delta (c,b,a).

A ***pivot-cycle*** $Pv(C_0,C_1)$ is a subgraph of G induced by $V(C_1)$, where $V(C_0) \subseteq V(C_1)$, $V(C_0)$ forms a closed odd pivot-path $C_0(p_0,p_1,\ldots,p_{2m})$, $m \geq 1$ in G, and $V(C_1)$ forms an anti-cycle C_1 in G such that for each vertex p_i in C_0: there is an anti-path aP_i connecting p_{i-1}, p_{i+1} in $N(p_i)$, operations +- mod $2m$+1, i.e., $C_1 = (aP_1,aP_3,\ldots,aP_0,aP_2,\ldots,aP_{2m})$. We have an ***asteroid*** $Ar(C_1,C_0)$ which is a subgraph of $\sim G$ induced by the vertex set $V(Pv)$ and corresponds to a pivot-cycle $Pv(C_0,C_1)$ in G. Hence "*G has a pivot-cycle*" is exactly equivalent to "*~G has an asteroid*".

We call the odd cycle C_0 of a pivot-cycle $Pv(C_0,C_1)$ the ***front-cycle*** of Pv and the anti-cycle C_1 the ***rear-cycle*** of Pv. Similarly an asteroid $Ar(C_1,C_0)$ has its front-cycle C_1 and odd rear-cycle C_0. *A rear-cycle* always implies an *anti-cycle*. We may call a vertex in the front-cycle C_0 of a pivot-cycle a ***pivot*** and the odd cycle C_0 itself a *pivot-cycle*. Similarly we may call the front-cycle C_1 of an asteroid Ar itself an *asteroid*. So it can be considered that there are just two cycles of the ***pivot-cycle*** $\boldsymbol{C_0}$ and ***asteroid*** $\boldsymbol{C_1}$.

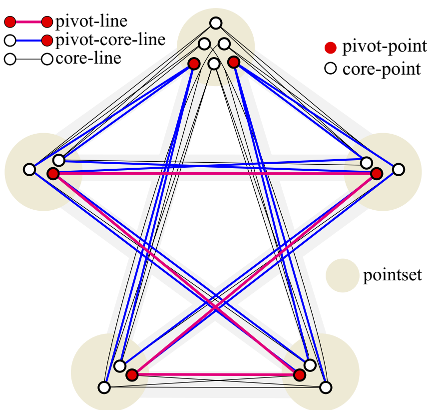


We call a pivot-cycle (C_0,C_1) / asteroid (C_1,C_0) a ***k-pivot-cycle*** / ***k-asteroid*** respectively, where $|C_0| = k$, k is odd. Similarly we call an OZ-cycle (Z_0,Z_1) a ***k-OZ-cycle*** such that $|Z_0| = |Z_1| = k$, k is odd. An induced subgraph of a graph G such as a k-OZ-cycle / k-pivot-cycle / k-asteroid is ***spanning*** if k equals the size of G. Note: There is no 3-OZ-cycle, i.e., OZ-triangle. We say a pivot-cycle / asteroid is ***elementary*** / ***simple*** according as the cycle C_0 is elementary / simple respectively. As well an OZ-cycle is ***elementary*** / ***simple*** according as the cycle Z_0 is elementary / simple respectively.



a pivot-cycle

The delta graph G_d of a graph G is a derived graph from G such that a vertex p of G_d is a delta (i.e., a 2-path) of G. A vertex pair (p_1,p_2) in G_d is joined iff p_1 and p_2 have a common edge in G and the edges of delta p_1, p_2 form a *3-path* in G. We call a vertex / edge of the delta graph G_d a ***point*** / ***line*** respectively. We make a *point set partition* Π of G_d such that for all vertices x in G: every point p which has the middle vertex x is in the same point subset $\Pi(x)$.



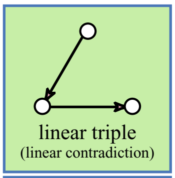
a delta graph

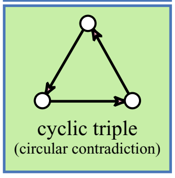
We call an element in Π a ***pointset*** and Π the ***pointset partition*** of delta graph G_d. A point of G_d corresponds to a delta in G and a pointset of Π in G_d corresponds to a vertex in G.

A point p of the delta graph G_d of G is a ***pivot-point*** if p is a pivot in G, otherwise a ***core-point***. Accordingly the delta graph G_d can be partitioned into two parts. We call the subgraph M_r / M_c induced by all of pivot-points / core-points of G_d the ***pivot-map*** / ***core-map*** respectively. Lines in a pivot-map / core-map are called ***pivot-lines*** / ***core-lines***. We make a directed graph D_p called the ***pivot digraph*** which contains all the points of the delta graph G_d, all the pivot-lines (i.e., edges of pivot-map M_p) and lines connecting the pivot-map M_p with the core-map M_c.

A directed path P is ***alternating*** if every two edges adjacent on P have an opposite orientation to each other. A directed path P is ***linear*** if every edge on P has the same orientation.

We call a property of a directed graph D ***transitivity*** such that whenever directed edges ab and bc are in D, an edge ac is in D for all vertex triple (a,b,c) of D. A directed graph D is ***transitively complete*** iff D is acyclic and has the transitivity property. The definition is valid for a case where D is a directed multigraph.

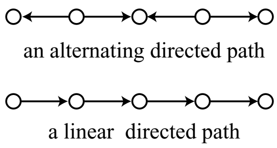


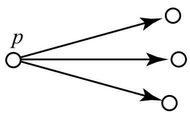


We say a vertex triple $T(a,b,c)$ of a directed graph D is in ***triple contradiction*** when T is (1)a cyclic triple, or (2)a linear triple (i.e., forms a linear 2-path, lacking the third edge). Besides we call the state of (1) ***circular contradiction*** and (2) ***linear contradiction*** on D.

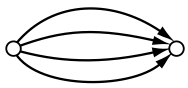
Edges e_1, e_2 of a directed graph D are ***coherent** for* a vertex v in D iff both of e_1 and e_2 are either incoming edges or outgoing edges of v. A directed multigraph D_m is ***coherent*** iff for each vertex pair in D_m: all edges joining the two vertices have the same orientation.

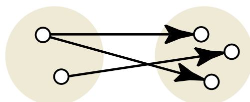
An ***orientation*** of an undirected graph G is a directed graph D obtained by giving an orientation to every edges of G. We call the orientation D of an undirected graph G a ***plotnikov digraph*** of G. A plotnikov digraph D is ***complete*** iff D is *transitively complete*. An undirected graph G is ***transitive*** iff it has a complete plotnikov digraph. Transitive graphs are also called ***comparability** graphs* [6] or ***transitively orientable** graphs* [9].





coherent edges for p

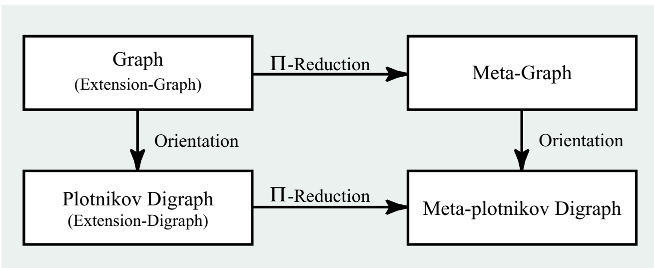

coherent multi-edges


coherent for independent set partition Π

A directed graph D has an independent set partition Π (like undirected graphs). If for all independent set pair (P,Q), $P,Q \in \Pi$: for all vertex pair (p,q), $p \in P$, $q \in Q$: each edges pq has the same orientation like $P \rightarrow Q$, we say edges of D are ***coherent for** Π*.

Suppose an undirected graph G_1, an independent set partition Π of G_1 and its plotnikov digraph D_1. We have a reduced graph G_0 from G_1 such that a vertex of G_0 is an element of Π and an edge of G_0 is reduced multi-edges of an independent set pair of Π. We call G_0 a ***meta-graph*** of G_1 and a plotnikov digraph D_0 of G_0 a ***meta-plotnikov digraph*** of G_1. We will call this graph operation ***Π-reduction***.

When a plotnikov digraph D_1 of G_1 is coherent for the independent set partition Π of G_1 and the orientation of the meta-plotnikov digraph D_0 of G_1 corresponds to the orientation of the elements of Π, we say the plotnikov digraph D_1 is ***coherent with*** the meta-plotnikov digraph D_0.



On the contrary, a graph G_1 is called an ***extension-graph*** of a graph G_0 when G_0 is a meta-graph of G_1. A plotnikov digraph D_1 of G_1 is called an ***extension-digraph*** of G_0. If the extension-digraph D_1 is *coherent with* the independent set partition Π of G_1, we say that the extension-digraph D_1 is ***coherent***. The *delta graph* G_d of a graph G is an extension-graph of G and a *plotnikov digraph* D_d of a delta graph G_d is an extension-digraph of G.

A transitive graph G_1 obtained from an intransitive graph G_0 by adding edges of an edge set α is called a ***supplement graph*** of G_0, and the edge set α (an edge subset of the complement of G_0) is called the ***supplement edge set***. We say supplement edge set α is ***minimal*** iff G_1 becomes intransitive by eliminating any edge of α.

A graph G is said to be **contradictious** *iff G and the complement are elementary OZ-cycles and have no smaller OZ-cycles.* An ***intransitive*** graph G is ***minimally intransitive*** iff G becomes transitive by removing any vertex of G. We say a graph G is ***strongly intransitive*** iff *both G and the complement* of *G are* ***minimally intransitive*** *and have* **a spanning pivot-cycle**.

A graph G is ***perfect*** iff the maximum clique size equals the chromatic number for all induced subgraphs of G. With respect to an arbitrary graph, maximum clique size = necessary minimum coloring number ≤ chromatic number = minimum independent set partition number. An imperfect graph is ***minimally imperfect*** iff it becomes perfect by removing any vertex of it. A graph G is ***berge*** iff G and the complement of G have neither odd holes nor odd antiholes.

Berge's Conjecture: *A graph G is perfect iff G is a berge graph .*

Theorems

T1: [*Perfect Graph Theorem*] *The complement of a perfect graph is perfect.*

T2: *An induced subgraph of a perfect graph is perfect.*

T3: *A perfect graph and its complement have no odd holes.*

T4: *When a graph G and its complement have no odd holes, any induced subgraph of G and its complement have no odd holes.*

T5: (Removed.)

T6: *A vertex induced subgraph of a transitively complete directed graph is transitively complete.*

T7: *An induced subgraph of a transitive graph is transitive.*

T8: *A chain of a complete plotnikov digraph D of a transitive graph G is a clique of G and an anti-chain of D is an independent set of G.*

T9: [Dilworth's Theorem] *The maximum anti-chain size of a partially ordered set P equals the minimum chain partition number and the longest chain length equals the minimum anti-chain partition number.*

T10: *A transitive graph is perfect.* (The converse is not true.)

T11: [*Triple Contradiction Theorem*] *A directed graph D is transitively complete iff every vertex triple of D is not in triple contradiction.*

T12: *A plotnikov digraph D is complete iff D has the transitivity property*.

T13: *When an OZ-cycle O_z and its complement have no odd holes, there exists at least such one short chord (i.e., an edge joining endpoints of a 2-path on the cycle) of O_z that adding the edge to O_z makes an even hole including the edge*. (Disproved by Stertenbrink.)

T14: [Algorithm A] *A graph G is transitive iff G has no pivot-cycles.* (A complete orientation algorithm for transitive graphs)

T15: [Algorithm B] *A complete supplemental plotnikov digraph of an arbitrary graph with minimal supplement edge set can be obtained in polynomial time*. ✯

T16: [Algorithm C] *If a graph G_0 and its complement has no odd holes, there exists a supplement graph G_1 of G_0 satisfying the inequality: maximum clique size of $G_1 \leq$ maximum clique size of G_0.* ✯

T17: [Berge's Conjecture] *A graph is perfect iff it is a berge graph.*

T18: *Odd holes and odd antiholes are contradictious graphs.*

T19: [Algorithm P] *A transitive graph G_1 has always a complete plotnikov digraph D_1 coherent with the meta-plotnikov digraph D_0 of G_1.* (A coloring algorithm for transitive graphs)

T20: *An arbitrary antihole of size > 4 is intransitive.*

T21: *Whenever a graph G has an OZ-cycle, G has a pivot-cycle*.

T22: *A graph G is intransitive if G has a pivot-cycle*.

T23: *A graph G is strongly intransitive if G is a contradictious graph*.

T24: *Odd holes and odd antiholes are strongly intransitive graphs* .

T25: *The following statements are equivalent for an undirected graph G.*

(1)*G is transitive.*

(2)*G has no OZ-cycles.*

(3)*G has no pivot-cycles.*

(4)*G has no elementary pivot-cycles.*

(5)~*G has no asteroid.*

(6)*The pivot-map of the delta graph of G is bipartite.*

T26: *A strongly intransitive graph is imperfect*. ✯

T27: *A strongly intransitive graph is a minimally imperfect graph*.

T28: *The maximum clique size and the chromatic number of a graph which has no strongly intransitive graphs are coincident*. ✯

T29: *A graph which has no strongly intransitive graphs is perfect*.
T30: *A strongly intransitive graph is either an odd hole or an odd antihole*. ✯
T31: *A graph is strongly intransitive iff it is a contradictious graph*. ✯
T32: *A graph is perfect iff it has no strongly intransitive graphs*.
T100-T125: (10 Theorems were here. They are all valid and have proofs but removed except T104, T123, T124, T125. Those are renumbered as T19, T20, T21, T22 respectively.)
T130: *A graph is bipartite iff all its elementary cycles are even*.
T131: *A graph is 2-colorable iff it has no odd elementary cycles*.
T132: *A bipartite graph is transitive*.
T133: *There is no common point in any triangles in the delta graph* G_d *of a graph G.* (A delta graph has no other cliques than triangles.)
T134: *The pointset partition* Π *of the delta graph* G_d *of a graph G is an independent set partition of* G_d.
T135-T136: (Removed.)
T137: *A graph G is eulerian iff the edge set of G can be partitioned into elementary cycles*.
T138: *A graph G has no pivot-cycles iff the pivot-map of the delta graph of G has no odd elementary cycles.*
T139: *The coupled edges of a pivot are coherent for its pivot-vertex*.
T140: *Given a 2-connected graph G, the delta graph* G_d *of G. An elementary path of length > 2 in G corresponds to an elementary path in* G_d *and an elementary path in* G_d *corresponds to a path in G*.
T141: *A graph G has an odd cycle iff G has an odd elementary cycle*.
T142: *If a 2-connected graph G has a complete and coherent extension-digraph, then G is transitive*.
T143: *A graph G has an odd simple cycle iff G has an odd elementary cycle*.
T144: *The following three statements are equivalent for a graph G*.
(1)*G has an odd elementary cycle*.
(2)*G has an odd simple cycle*.
(3)*G has an odd cycle*.
T145: *Given a 2-connected graph G, the delta graph* G_d *of G, the pointset partition* Π *of* G_d*. A vertex of G one to one corresponds to a pointset in* Π *and one to many corresponds to points in* G_d*. An edge in G one to many corresponds to lines in* G_d.
T146: *Given a graph G, the delta graph* G_d *of G, the pointset partition* Π *of* G_d*. Suppose points* $p_0,p_1 \in P$, $q_0,q_1 \in Q$, $P,Q \in \Pi$*. Whenever there exist lines* p_0q_0, p_1q_1 *in* G_d*, the lines* p_0q_1, p_1q_0 *exist in* G_d.
T147: *Given a graph G, the delta graph* G_d *of G, the pointset partition* Π *of* G_d*, the pivot-map* M_p*(core-map* M_c*) of* G_d*. A connected component* C_p *of* $M_p(M_c)$ *corresponds to a set S of the pointset pairs of* Π *and no other components than* C_p *have pivot-lines(core-lines) belonging to a pointset pair* $\in$ *S.*
T148: *A subgraph of a graph G induced by the vertices of a core-path in G is a twisted path in G*.
T149: *A core-line of the delta graph* G_d *of size > 2 of a graph G has corresponding triangles in G*.
T150: *A vertex pair (p,q) in a strongly twisted path* T_p *is always joined*.
T151: (Removed.)
T152: *A path in the core-map of the delta graph of a graph G corresponds to a twisted path in G*.
T160: *A graph G has a pivot-cycle iff G has an OZ-cycle*.
T161: *A graph G has a pivot-cycle iff G has an elementary pivot-cycle*.
T170: *For every berge graph* G_0*, there exists a perfect graph* G_1 *obtained by adding edges to* G_0 *satisfying the following inequality: maximum clique size of* $G_1 \leq$ *maximum clique size of* G_0. ✯
T171: *Every vertex induced subgraph of a berge graph is a berge graph.*
T172: *A perfect graph is a berge graph.*

Proofs

T1: [Perfect Graph Theorem]
The complement of a perfect graph is perfect.

Proof: See Lovász, 1972 [16]. □

T2: *An induced subgraph of a perfect graph is perfect*.

Proof: By definition if a graph G is perfect, the maximum clique size equals the chromatic number for all induced subgraphs of G. Let G_s be an induced subgraph of G. Since a subgraph of G_s is a subgraph of G, for all subgraphs of G_s: the maximum clique size equals the chromatic number. Hence every induced subgraph of a perfect graph G is perfect. □

T3: *A perfect graph and its complement have no odd holes*.

Proof: An odd elementary cycle C_o is not perfect because, the maximum clique size of C_o is 2 and its chromatic number is 3, hence an odd elementary cycle is imperfect. Assume a perfect graph G has an odd elementary cycle C_o. By T2, C_o comes to be perfect. This is a contradiction. Hence a perfect graph has no odd holes. Assume that the complement aG of a perfect graph G has an odd hole. By T1, the complement aG of a perfect graph G is perfect. Then it turns that perfect graph aG has an odd hole. This contradicts the above. Hence the complement of a perfect graph also has not an odd hole, i.e., a perfect graph and its complement have no odd holes □

T4: *When a graph G and its complement have no odd holes, any induced subgraph of G and its complement have no odd holes*.

Proof: We prove the contraposition of the theorem, i.e., when a subgraph G_s of G or its complement aG_s have an odd hole, the graph G or its complement aG have an odd hole. It is obvious that when G_s has an odd hole, G has an odd hole. As well, when aG_s has an odd hole, aG has an odd hole. □

T6: *A vertex induced subgraph of a transitively complete directed graph is transitively complete.*

Proof: By definition, a transitively complete directed graph D is acyclic and whenever edges ab, bc are in D, an edge ac is in D for all vertex triple (a,b,c). Since D has no cycles, a subgraph D_s of D also has no cycles, hence D_s is acyclic. Moreover for all triples of the induced subgraph D_s: transitive triple relations are always valid. Therefore every induced subgraph of a transitively complete directed graph is transitively complete. □

T7: *An induced subgraph of a transitive graph is transitive*.

Proof: Immediate from T6. □

T8: *A chain of a complete plotnikov digraph D of a transitive graph G is a clique of G and an anti-chain of D is an independent set of G.*

Proof: A chain C_h is a linear elementary path $(p_0,p_1,..,p_m)$ of a directed graph D. As D is complete, whenever edges p_0p_{k-1}, $p_{k-1}p_k$ exist, an edge p_0p_k exists in *D*. Accordingly as we have edges p_0p_1 and p_1p_2, we also have edges $p_0p_2,\ldots, p_0p_{k-1}, p_0p_k$. That is, when there is an oriented path $(p_0,p_1,..,p_k)$ in *D*, there is an edge connecting the end vertices p_0 and p_k in *D*. Hence for all vertex pair (p,q) in C_h: there is an oriented path (p,q) or (q,p) is in *D* and the edge *pq* or *qp* is in *D*. Consequently a subgraph of *G* induced by the vertices of C_h is a clique of *G*. An anti-chain aC_h is a vertex set of *D* with no edges connecting the vertices in aC_h. Therefore a subgraph induced by aC_h is an independent set of *G*. □

T9: [Dilworth's Theorem]
The maximum anti-chain size of a partially ordered set P equals the minimum chain partition number and the longest chain length equals the minimum anti-chain partition number.

Proof: See Mirsky, 1971 [18], Liu, 1985 [15]. □

T10: *A transitive graph is perfect*. (The converse is not true.)

Proof: By T7, every induced subgraph of a transitive graph is transitive. Then we only need that the maximum clique size equals the chromatic number for a transitive graph. Let *D* be a complete plotnikov digraph of a transitive graph *G*. By T9, the longest chain length of *D* equals the minimum anti-chain partition number. By T8 a chain of *D* is a clique of *G* and an anti-chain of *D* is an independent set of *G*, hence the maximum clique size of *G* equals the minimum independent set partition number. Thus a transitive graph is perfect. □

T11: [Triple Contradiction Theorem]
A directed graph D is transitively complete iff every vertex triple of D is not in triple contradiction.

Proof: By definition, a transitively complete directed graph *D* is acyclic and whenever edges *ab*, *bc* are in *D*, an edge *ac* is in *D* for all vertex triple (a,b,c). Since *D* is acyclic, cyclic triples are not in *D*. Assume a linear triple (a,b,c) is in *D*, i.e., there are edges *ab*, *bc* but *ac* is not. This contradicts the transitivity of *D*. Hence transitively complete directed graph has no triple contradiction. Assume there is no triple contradiction for all triples in *D* and *D* has an oriented cycle $C(p_1,p_2,..,p_k)$. Since there is no linear contradiction, when edges $p_1p_2, p_2p_3,.., p_kp_1$ exist, edges $p_3p_5,.., p_3p_k, p_3p_1$ must exist. Therefore it turns that a cyclic triple (p_3,p_1,p_2) exists. Contradiction. Hence if no triple contradiction on *D*, then *D* is acyclic. The condition that linear contradiction does not exist satisfies the transitivity of *D* because it implies the status whenever edges *ab*, *bc* exist, an edge *ac* exists. Accordingly the assertion that every vertex triple of *D* is not in triple contradiction is the necessary and sufficient condition for *D* is transitively complete. □

T12: *A plotnikov digraph D is complete iff D has the transitivity property*.

Proof: By definition a complete plotnikov digraph *D* has the transitivity property. To prove the converse, it is enough to show that if *D* has the transitivity property, then *D* is acyclic. Since plotnikov digraph is an orientation of an undirected graph, there are neither loops nor parallel-edges. Assume *D* has an oriented cycle $C(p_0,p_1,...,p_m)$. By hypothesis, whenever directed edges *ab* and *bc* are in *D*, an edge *ac* is in *D*. Hence there are edges $p_0p_2, p_0p_3,..., p_0p_m$, while the edge p_mp_0 exists in the cycle C. As the edges p_0p_m and p_mp_0 are parallel-edges, it contradicts the definition of plotnikov digraph. Consequently *D* has no oriented cycles, and then *D* is acyclic and complete. □

T13: *When an OZ-cycle O_z and its complement have no odd holes, there exists at least such one short* *chord (i.e., an edge joining endpoints of a 2-path on the cycle) of O_z that adding the edge to O_z makes an even hole including the edge*. (Disproved by Stertenbrink.)

Counter Example: 9 vertices, 17 edges
(1,2),(1,4),(1,5),(1,6),(1,7),(1,9),(2,3),(2,5),(2,6),(3,4),(3,6),(4,5),(5,6),(6,7),(6,9),(7,8),(8,9)

T14: [Algorithm A] *A graph G is transitive iff G has no pivot-cycles*. (A complete orientation algorithm for transitive graphs)

Proof: By T22, a graph *G* with a pivot-cycle is intransitive. Then we prove the converse that a graph *G* with no pivot-cycle is transitive. T142 says that if a graph *G* has a complete and coherent extension-digraph, then *G* is transitive. So if the delta graph G_d of *G* with no pivot-cycles is transitive and if we could get a complete and coherent plotnikov digraph of G_d, we have done. However this is impossible from T130 which says that there is no common point in any triangles in the delta graph G_d. This means that almost every triangle in G_d forms so called a *net*. A net is a graph which consists of a triangle (*a*,*b*,*c*) and edges *ax*, *by*, *cz*. It is easily ascertained that a net is intransitive. To avoid such a bad configuration, we omit all of core-lines in G_d and make a complete pivot digraph D_p first.

A pivot digraph is a digraph contains all of pivot-lines and the lines connecting the pivot-map with the core-map. After we complete the orientation of the pivot digraph D_p, we make up the orientation for the core-map M_c in the last stage and get a complete plotnikov digraph P_d of *G*. By T137, all the lines in the core-map correspond to triangles. That is, there is no linear triples related to the core-map. So by T11, the only constraint condition of the orientation for the core-map is just to avoid circulation and it is easily done. We show that if the pivot-map of the delta graph G_d has no odd elementary cycles, a complete plotnikov digraph P_d of *G* is to be constructed by the following algorithm. Without loss of generality, we assume *G* is 2-connected and the size of *G* is greater than five.

[Algorithm A]

(1)Given a graph $G(V,E)$. $|V| = n$, $|E| = m$.
(2)Make the delta graph G_d of *G*, the pointset partition Π of G_d, the pivot-map M_p, the core-map M_c.
(3)Make a pivot digraph $D_p(V_p,E_p)$, $V_p = V(G_d)$, $E_p = E(G_d) - E(M_c)$.
(4)Get the connected components *Cp* of the subgraph induced by $V(M_p)$ in D_p.
(5)Pick an arbitrary component $C_p \in Cp$.
  Get a 2-coloring of C_p. If C_p is not 2-colorable end.
(6)Find a predetermined line *e* in C_p and decide the color orientation.
  For all lines of the component C_p: [Set Orientation] of each line by the color orientation.
(7)For all undetermined lines *pq* connecting C_p with the core-map M_c, *p* in C_p, *q* in M_c:
  If there is a predetermined line *e* in D_p incident with *p*, let the orientation of *e* be *O*,
  else an arbitrary orientation be *O*.
  [Set Orientation] of the line *pq* by the orientation *O*.
(8)Continue (5) until connected components *Cp* becomes empty.
(9)Make a directed graph $P_d(V,E)$.
  Copy the pointset pair orientation of Π in D_p to P_d.
  Make the partial order set O_p from P_d.
(9)Transform the partial order O_p to a total order O_t.
  For all undetermined edges *e* in P_d: Set the orientation according to the total order O_t.

(10) P_d is a complete plotnikov digraph of G.

[Set Orientation]
(1)Given a line pq in the pivot digraph D_p, a line orientation O_r.
(2)Set the orientation O_r to the line pq in D_p.
(3)For all undetermined lines e of the same pointset pair of the edge pq:
Set the orientation O_r to the line e.
(4)Let P,Q,R be pointsets of Π, $p \in P$, $q \in Q$, $P,Q,R \in \Pi$.
For all pointset triangles (P, Q, R) including the line pq:
If the pointset pair orientation (Q,R) / (P,R) is determined and the 2-path (P,Q,R) / (R,P,Q) is linear, and the pointset pair orientation (P,R) / (Q,R) is undetermined, decide the orientation O_s avoiding circular-contradiction, and for all undetermined lines e of the pointset pair (P,R) / (Q,R):
set the orientation O_s to the line e.

First we construct the delta graph G_d of G, the pointset partition Π of G_d, the pivot-map M_p, and the core-map M_c. The time complexity of step (2) is polynomial for the points number of G_d = the number of the edge pairs in $G \leq m(n\text{-}2) = n(n\text{-}1)(n\text{-}2) / 2$, i.e., at most $O(n^3)$. Step (3), we make the pivot digraph D_p having all the points of G_d. D_p contains all of lines of the pivot-map M_p and all of lines connecting the pivot-map M_p with the core-map M_c. At step (4) we divide the pivot-map M_p into the connected components Cp in D_p. Step (5) pick an arbitrary connected component $C_p \in Cp$ and get the 2-coloring of C_p. If C_p is not 2-colorable, M_p is not 2-colorable, and by T131 M_p has an odd elementary cycle and by T138, T22, G is intransitive. We can get a 2-coloring of an arbitrary 2-colorable graph by a simple breadth-first-search method. Step (6) set the orientation of lines in the component C_p according to the end points color(0) / (1). Since all lines of each pointset pair must be coherent, we set the orientation of all lines of each pointset pair in a bundle by the orientation of the first determined line.

Accordingly it is probable that some lines in C_p are predetermined at some preceding stage. Therefore first we seek a predetermined line in the component C_p. If a predetermined line $p_0 \rightarrow q_0$ exists, we decide the color orientation like color(p_0)$\rightarrow$color(q_0), else we use the default orientation color(0)$\rightarrow$color(1). It is probable that two points in C_p have different colors and contained in a same pointset. This is a contradiction but we show it does not happen. Assume points p_0 / q_0 in C_p have color(0) / (1) respectively and p_0, q_0 are in the same pointset. As C_p is connected, there is an odd elementary path $P(p_0,p_1,..,p_{2m},q_0)$ connecting p_0 with q_0 and there must be an line q_0q_1 doubled over the line p_0p_1. There are two topological cases, (1) $P = (p_0,p_1,..,p_{2m},q_0,q_1)$, (2) $P = (p_0,p_1,..,p_{2m\text{-}1},q_1,q_0)$. Since the point pair (p_0,q_0) / (p_1,q_1) are in a same pointset of Π respectively and there are lines p_0p_1, q_0q_1 in D_p, by T146, we have lines p_0q_1, p_1q_0 in D_p. For the first case, the path $(q_0,p_1,..,p_{2m})$ is closed and forms an odd elementary cycle. As well for the second case, the path $(p_1,..,p_{2m\text{-}1},q_1,q_0)$ is closed and forms an odd elementary cycle. These contradict the hypothesis.

The function [Set Orientation] sets the orientation of an undetermined line pq. All of the lines of the pointset pair, which contains the line pq, are set by the same orientation in a bundle. If the line pq is in a pointset triangle (P,Q,R) and the orientation of the coupled pointset pairs forms a linear 2-path, we give a forced orientation to the third pointset pair to avoid the circulation. Note: Since C_p has no odd cycles, there is no triangle in C_p. Further from T133 we know that any two triangles in the delta graph G_d have no common lines, and from T147 that a connected component C_p dominates the orientation of a pointset pair. Accordingly we do not call [set orientation] recursively.

Step (7) set the orientation of lines connecting the component C_p of the pivot-map M_p with the core-map M_c. Since every point $p(x,y,z)$ in M_p is a pivot-point, by T139, lines incident with p must be coherent. If there is a line e incident with p and predetermined in D_p, then the orientation of e is applied to the other lines incident with p. If there is no predetermined line, an arbitrary orientation is given to the first line. At step (8) we complete the orientation of D_p, where the lines of D_p are coherent with the pointset partition Π. Step (9) We reduce the pivot digraph D_p to a directed graph P_d.

All of points in a pointset of D_p is contracted to a vertex of P_d and all of lines in a pointset pair in Π is reduced to an edge of P_d. P_d has the same number of vertices / edges of G. The determined line orientations of D_p are all bundled (i.e., coherent) with each pointset pair orientation. So we can copy the pointset pair orientations to P_d. We consider the partial order set O_p which consists of all of vertices and all of determined edges in P_d. We can transform the partial order O_p to a total order O_t using some topological sorting method. (See Knuth, 1994 [12].) The only constraint condition for the orientation is being acyclic. And it is to be fulfilled by deciding the orientation according to the total order O_t, i.e., just give an orientation to each undetermined edge pq like if $p < q$ in the total order O_t then $p \to q$ else $q \to p$.

We decide every lines in the pivot digraph D_p just once. Accordingly the total complexity of *algorithm A* is proportional to the number of lines in D_p, i.e, at most $O(Cn^6)$, where C is the orientation cost per line. Costs for making connected components, 2-coloring, topological sorting are polynomial-time respectively. Thus if the pivot-map M_p of G_d has no odd elementary cycles, we can obtain a complete plotnikov digraph P_d of G by *algorithm A* in polynomial time. Hence if the pivot-map M_p of the delta graph G_d has no odd elementary cycles, G is transitive. Consequently by T138, a graph with no pivot-cycles is transitive. Above all, the statement holds. □

T15☆ [Algorithm B] *A complete supplemental plotnikov digraph of an arbitrary graph G with minimal supplement edge set can be obtained in polynomial time*.

Proof: Pending...

T16☆ [Algorithm C] *If a graph G_0 and its complement has no odd holes, there exists a supplement graph G_1 of G_0 satisfying the inequality: maximum clique size of $G_1 \leq$ maximum clique size of G_0.*

Proof: Pending...

T17: [Berge's Conjecture]
A graph is perfect iff it is a berge graph.

Proof 1: By T172 a perfect graph is a berge graph. Then we will prove that if a graph G and the complement aG have no odd holes, G is perfect. When a graph G_0 and its complement aG_0 have no odd holes, by T16 we have an supplement graph G_1 of G_0 such that,

maximum clique size of $G_1 \leq$ maximum clique size of G_0

As G_1 is obtained by adding edges to G_0, the chromatic number of G_1 is always larger or equals the chromatic number of G_0. Hence,

maximum clique size of $G_0 \leq$ chromatic number of G_0

chromatic number of $G_0 \leq$ chromatic number of G_1

Since G_1 is transitive, G_1 is perfect by T10 and

maximum clique size of G_1 = chromatic number of G_1.

Then all the equalities hold in the inequalities above. Hence,

maximum clique size of G_0 = chromatic number of G_0

By T4, when a graph G and its complement have no odd holes, any induced subgraph of G and its complement have no odd holes. Therefore these equalities hold for each subgraph G_0 of G, then if graph G and the complement have no odd holes, G is perfect. By definition, a berge graph and the complement have no odd holes, hence a berge graph is perfect. We have done. □

Proof 2: By T32, a graph is perfect iff it has no strongly intransitive graphs and by T30, a strongly intransitive graph is either an odd hole or an odd antihole. Hence a graph is perfect iff it has neither odd holes nor odd antiholes. Let us show that a graph which has neither odd holes nor odd antiholes is equivalent to a berge graph. By definition a berge graph has neither odd holes nor odd antiholes. Consider a graph G which has neither odd holes nor antiholes. Assume the complement aG of G has either an odd hole H or an odd antihole aH. Then it turns that G has either an odd antihole $\sim H$ or an odd hole $\sim aH$. This contradicts the hypothesis. Hence a graph which has neither odd holes nor odd antiholes is a berge graph. Therefore a graph G is perfect iff G is a berge graph. □

Proof 3: By T172 a perfect graph is a berge graph. Then we prove that a berge graph is perfect. By T170, a berge graph B_0 has a perfect supplement graph B_1 satisfying the inequalities below.

maximum clique size of $B_1 \leq$ maximum clique size of B_0 (1)

As B_1 is obtained by adding edges to B_0, the chromatic number of B_1 is always larger or equals the chromatic number of B_0. Hence,

maximum clique size of $B_0 \leq$ chromatic number of B_0 (2)
chromatic number of $B_0 \leq$ chromatic number of B_1 (3)

By hypothesis, B_1 is perfect. Hence by the definition of perfect graphs,

maximum clique size of B_1 = chromatic number of B_1 (4)

Consequently, all the equalities hold in the inequalities (1)-(3). Hence,

maximum clique size of B_0 = chromatic number of B_0 (5)

By T171, when a graph G is a berge graph, every subgraph of G is a berge graph. Accordingly equality (5) holds for all induced subgraphs of G and then G is a perfect graph. Hence the assertion holds. □

T18: *Odd holes and odd antiholes are contradictious graphs.*

Proof: By definition, an OZ-cycle Z must have the front-cycle Z_0 and the rear-cycle Z_1 of the alternate vertex sequence of Z_0. Obviously an odd cycle C_o, $|C_o| = 2m+1$, $m > 1$ is an elementary OZ-cycle for C_o has an elementary spanning cycle $C_o(p_0,p_1,..,p_{2m})$ and a spanning anti-cycle $C_1(p_0,p_2,..,p_{2m-1})$ of the alternate vertex sequence of C_o. Apparently the odd cycle C_o has no more OZ-cycles.

For the complement aC_o of the odd cycle C_o, We have the other spanning cycles A_0, A_1. Start at the vertex p_0 on C_o and connect m-th vertices continuously, we obtain an elementary spanning anti-cycle $A_0(p_0,p_m,p_{2m},p_{m-1},p_{2m-1},...,p_1,p_{m+1})$. This is the front-cycle A_0 of aC_o. Actually the alternate vertex sequence of A_0 forms a spanning cycle $A_1(p_0,p_{2m},p_{2m-1}..,p_1)$ and A_1 exactly corresponds with C_o. Thus A_1 is the rear-cycle of aC_o and the complement aC_o of the odd cycle C_o is an elementary OZ-cycle. The rear-cycle A_1 of aC_o is the odd cycle C_o itself and there is no other candidate to be the rear-cycle of aC_o, hence aC_o can not have any other smaller OZ-cycles than (A_0,A_1). Accordingly an odd cycle (except triangle) and its complement are both elementary OZ-cycles and have no smaller OZ-cycles, then these are contradictious cycles. □

T19: [Algorithm P] *A transitive graph G_1 has always a complete plotnikov digraph D_1 coherent with the meta-plotnikov digraph D_0 of G_1.* (A coloring algorithm for transitive graphs)

Proof: We call a vertex of a directed graph who has no incoming / outgoing edges a *source* / *sink* respectively. As G_1 is transitive, we have a complete plotnikov digraph D_1 of G_1. We will get a minimum independent set partition Π of G_1 and make D_1 coherent with the meta-plotnikov digraph D_0 of G_1 by the following algorithm.

[Algorithm P]
1. Given a transitive graph G_1, a complete plotnikov digraph D_1 of G_1.
2. Copy D_1 to D. $i = 1$.
3. Move all of the sources of D into the vertex set P_i. Increment i.
4. Continue (3) until D becomes empty.
5. Obtained set $\Pi = \{P_i\}$ is a minimum independent set partition of G_1.
   Let G_0 be the meta-graph of G_1 reduced by Π and a plotnikov digraph of G_0 be D_0.
6. Set the edge orientation of D_0 according to the index order of Π:
   if $i < j$ then $P_i \rightarrow P_j$ for all i,j , $1 \leq i,j \leq |\Pi|$.
7. The plotnikov digraph D_1 is coherent with the meta-plotnikov digraph D_0.

Since a complete plotnikov digraph D_1 is acyclic and by T6 an induced subgraph of D_1 is complete (then acyclic), there are always some sources at step (2). Further $|\Pi|$ = the longest chain length of D_1 because, as we at first picked all sources of D_1 and repeated the step, the steps count exactly corresponds to the longest chain length. As D_1 is complete, by T9 the longest chain length of D_1 equals its minimum anti-chain partition number, and by T8 the minimum anti-chain partition of D_1 is identical with the minimum independent set partition of G_1. Therefore Π is the minimum independent set partition of G_1. From the description of step (6), it is observed that D_1 is coherent with D_0. Thus we obtained a complete and coherent plotnikov digraph D_1 with the meta-plotnikov digraph D_0 of G_1. (The algorithm gives a minimum independent set partition Π, i.e., a coloring algorithm of transitive graphs.) □

T20: *An arbitrary antihole of size > 4 is intransitive.*

Proof: A 5-antihole, i.e., $\sim C_5 = C_5$ is intransitive. Then we assume that the size of an antihole aH is more than 5. Let the antihole of a hole H be $aH(p_0,p_1,p_2,p_3,p_4,..,p_n)$. Vertex pairs (p_0,p_1), (p_1,p_2),.., (p_n,p_0) are not joined in aH (they are the edges of H) and all other vertex pairs of aH are joined. Consider $C_5(p_0,p_2,p_4,p_1,p_3)$ in aH. As edges p_0p_1, p_1p_2, p_2p_3, p_3p_4 are not in aH, deltas (p_1,p_3,p_0), (p_2,p_4,p_1), (p_2,p_0,p_3), (p_4,p_1,p_3) are pivots. As edges p_4p_5, p_5p_6,.., p_np_0 are not in aH, the subgraph induced by $\{p_4,p_5,...,p_n,p_0\}$ is anti-connected. Hence $F(p_2,\{p_4,p_5,...,p_0\})$ is a fan and the delta (p_0,p_2,p_4) is a pivot. Consequently $C_5(p_0,p_2,p_4,p_1,p_3)$ is a pivot-cycle and then by T22, G is not transitive. □

T21: *Whenever a graph G has an OZ-cycle, G has a pivot-cycle*.

Proof: By definition, an OZ-cycle $Z(Z_0,Z_1)$ has an odd front-cycle $Z_0(p_0,p_1,..,p_{2m})$, $m > 1$ and a rear-cycle $Z_1(p_0,p_2,.., p_{2m-1})$ of the alternate vertex sequence of Z_0. If the rear-cycle Z_1 does not contain an anti-loop, i.e., for all i, $0 \le i \le 2m$: $p_{i-1} \neq p_{i+1}$, then there is always an anti-edge $p_{i-1}p_{i+1}$ in Z_1, and every 2-path (p_{i-1},p_i,p_{i+1}) is a pivot. Hence the OZ-cycle Z is a pivot-cycle (Z_0,Z_1). Whenever there is a loop pp in the rear-cycle Z_1 of Z, there are parallel-edges pq and qp in the front-cycle Z_0. In this case, we can remove the loop pp and one occurrence of the vertex p from Z_1 and remove both edges of pq and qp from Z_0. This operation doesn't change the parity of the front-cycle Z_0 and the rear-cycle Z_1 still remains as an anti-cycle. Thus we can always get a pivot-cycle $Pv(C_0,C_1)$ from OZ-cycle $Z(Z_0,Z_1)$ even if it has loops, where

the length of C_1 = the length of Z_1 - k,
the length of C_0 = the length of Z_0 - $2k$ = odd,
k = the number of loops in the rear cycle Z_1.

Hence if G has an OZ-cycle, G has always a pivot-cycle. □

T22: *A graph G is intransitive if G has a pivot-cycle*.

Proof: Assume that a graph G has a pivot-cycle C_p. By definition every delta in C_p is a pivot and then by T139, every pair of adjacent edges on C_p must be coherent in any complete plotnikov digraph. I.e., the path on C_p must be alternating. However it is impossible as the vertex number of the cycle C_p is odd and then a linear contradiction is inevitable on C_p. Hence by T11, G is intransitive. □

T23: *A graph G is strongly intransitive if G is a contradictious graph*.

Proof: By definition a contradictious graph G and the complement are elementary OZ-cycles and have no smaller OZ-cycles. Then by T160, G and the complement of G have no smaller pivot-cycles, hence by T14, those are minimally intransitive graphs. Since a contradictious graph is an elementary OZ-cycle and has no smaller OZ-cycles, the OZ-cycles of both G and the complement are spanning and have no loops. Therefor it is sure that G and the complement have spanning pivot-cycles corresponding to the OZ-cycles. Hence a contradictious graph is strongly intransitive. □

T24: *Odd holes and odd antiholes are strongly intransitive graphs* .

Proof: By T18, odd holes and odd antiholes are contradictious graphs, then by T23 strongly intransitive. □

T25: *The following statements are equivalent for an undirected graph G.*
(1) *G is transitive.*
(2) *G has no OZ-cycles.*
(3) *G has no pivot-cycles.*

(4) *G has no elementary pivot-cycles.*
(5) *~G has no asteroid.*
(6) *The pivot-map of the delta graph of G is bipartite.*

Proof: By T14 (3) is equivalent to (1). (2) and (3) are equivalent by T160. By definition (5) is equivalent to (3). By T138 and T130 (6) is equivalent to (3). By T161 (3) is equivalent to (4). □

T26✯ *A strongly intransitive graph is imperfect*.

Proof: Pending...

T27: *A strongly intransitive graph is a minimally imperfect graph*.

Proof: By T26 a strongly intransitive graph is imperfect. By definition, a strongly intransitive graph *G* is minimally intransitive, then every proper induced subgraph of *G* is transitive. As a transitive graph is perfect by T10, every proper subgraph of *G* is perfect. Hence strongly intransitive graph *G* is a minimally imperfect graph. □

T28✯ *The maximum clique size and the chromatic number of a graph which has no strongly intransitive graphs are coincident*.

Proof: Pending...

T29: *A graph which has no strongly intransitive graphs is perfect*.

Proof: Suppose a graph *G* which has no strongly intransitive graphs. It is obvious that every induced subgraph G_1 of *G* has no strongly intransitive graphs because, if G_1 has a strongly intransitive graph G_2, it turns that *G* has a strongly intransitive graph G_2. This contradicts the hypothesis. Hence every subgraph G_1 of *G* has no strongly intransitive graphs, then by T28 the maximum clique size and the chromatic number are equal for all induced subgraphs of *G*. Accordingly *G* is perfect. □

T30✯ *A strongly intransitive graph is either an odd hole or an odd antihole*.

Proof: Pending...

T31✯ *A graph is strongly intransitive iff it is a contradictious graph*.

Proof: Pending...

T32: *A graph is perfect iff it has no strongly intransitive graphs*.

Proof: By T29, if a graph *G* has no strongly intransitive graphs, *G* is perfect. By T26 a strongly intransitive graph is imperfect and from T2 a perfect graph has no imperfect subgraphs, hence a perfect graph *G* has not a strongly intransitive graph. □

T130: *A graph is bipartite iff all its elementary cycles are even*.

Proof: See König, 1936 [14]. □

T131: *A graph is 2-colorable iff it has no odd elementary cycles*.

Proof: A graph $G(V,E)$ is bipartite if V can be partitioned into two subsets V_0 and V_1 such that every edge of G joins a vertex of V_0 with a vertex of V_1. If a graph G is bipartite, we can give color(0) to V_0 and color(1) to V_1. This is a 2-coloring of G. As well it is self-evident that if a graph G is 2-colorable G is bipartite. Hence *bipartite* is equivalent to *2-colorable*. Since the right statements of T130 and T131 are equivalent, "G has no odd elementary cycles" is equivalent to "G is 2-colorable". □

T132: *A bipartite graph is transitive*.

Proof: By T130, T131 a bipartite graph G has a 2-coloring. Let a plotnikov digraph of G be P_d. Every edge of G has end vertices, one is painted color(0) and another color(1). We can set all the orientations in P_d as color(0)→color(1). It is obvious that there is neither a cyclic triple nor a linear triple in P_d. Hence by T11, P_d is complete and G is transitive. □

T133: *There is no common point in any triangles in the delta graph G_d of a graph G*. (A delta graph has no other cliques than triangles.)

Proof: Consider a triangle $T(a,b,c)$ of the delta graph G_d, where a,b,c are points $a(p_1,p_2,p_3)$, $b(p_2,p_3,p_4)$, $c(p_3,p_4,p_5)$ of G_d and p_i is a vertex in G. T forms an edge sequence p_1p_2, p_2p_3, p_3,p_4, p_4p_5. As T is a triangle, this sequence must be closed. Therefore the edge p_1p_2 must coincide with p_4p_5. I.e., $p_1 = p_4$ and $p_2 = p_5$ are required. Hence $T(a,b,c)$ consists of three points $a(p_1,p_2,p_3)$, $b(p_2,p_3,p_1)$, $c(p_3,p_1,p_2)$. Thus a triangle $T(a,b,c)$ of G_d exactly corresponds to a triangle (p_1,p_2,p_3) of G. Consequently there is no common point in any two triangles in the delta graph G_d of G. □

T134: *The pointset partition Π of the delta graph G_d of a graph G is an independent set partition of G_d.*

Proof: By definition, a point in G_d is a vertex of G_d and the pointset partition Π is a vertex set partition of G_d such that every point $P(s,t,u)$ in a pointset $\Pi(t)$ has the same middle vertex t in G. Let arbitrary two points in a pointset $\Pi(t)$ be $P_0(s_0,t,u_0)$, $P_1(s_1,t,u_1)$. Assume that there is a line P_0P_1 in G_d. By definition, points P_0, P_1 must have a common edge in G. Hence edge $tu_0 = s_1t$. This means $t = s_1$, $u_0 = t$ and it turns to $P_0 = (s_0,t,t)$ and $P_1 = (t,t,u_1)$. Contradiction. Consequently every point pair (P_0,P_1) in a same pointset of Π is not joined. Therefore a pointset in Π is an independent set of G_d and Π is an independent set partition of G_d. □

T137: *A graph G is eulerian iff the edge set of G can be partitioned into elementary cycles* .

Proof: See Harary, 1969 [8]. □

T138: *A graph G has no pivot-cycles iff the pivot-map of the delta graph of G has no odd elementary cycles*.

Proof: Let M_p be the pivot-map of the delta graph G_d of G. Without loss of generality we assume that G is 2-connected. By T140, an elementary path P of length > 2 of G corresponds to an elementary path in G_d. Accordingly a path in G corresponds to a path in G_d. By definition, every point in M_p is a pivot-point and every delta on a pivot-path in G has a corresponding point in M_p. Hence when G has a pivot-cycle, there is a corresponding odd cycle in M_p. By T144, when M_p has an odd cycle, M_p has an odd elementary cycle. Consequently if G has a pivot-cycle, the pivot-map M_p has an odd elementary cycle. By T140, when M_p has an odd elementary cycle, there is an corresponding odd cycle C_o in G. By definition every point p in M_p is a pivot-point and its corresponding delta is a pivot. Hence C_o is a pivot-cycle of G. Consequently the statement "*a graph G has a pivot-cycle*" is equivalent to "*the pivot-map M_p has an odd elementary cycle*". □

T139: *The coupled edges of a pivot are coherent for its pivot-vertex*.

Proof: When a vertex triple (a,b,c) is a linear triple (i.e., edges ab, bc exist and ac does not), by T11 coupled edges ab and bc must be coherent for the middle vertex b in any complete plotnikov digraph. Suppose a delta (q,r,s) in G is a pivot and r is the pivot-vertex. By definition, there is an anti-path $aP(q,a_1,a_2,..,a_k,s)$, where vertices $a_1,a_2,..,a_k$ are the neighbors of r. As the edges on aP are not in G, the deltas (q,r,a_1), (a_1,r,a_2),.., (a_k,r,s) are linear triples. Hence concerning above, edge pairs (qr,ra_1),(a_1r, ra_2),..., (a_kr,rs) must be coherent for the vertex r respectively in any complete plotnikov digraph. Therefore it is immediate that the coupled edges qr, rs are coherent for the pivot-vertex r. □

T140: *Given a 2-connected graph G, the delta graph G_d of G. An elementary path of length > 2 in G corresponds to an elementary path in G_d and an elementary path in G_d corresponds to a path in G*.

Proof: By T145 a vertex of G one to many corresponds to points in G_d. As well an edge in G one to many corresponds to lines in G_d. Without loss of generality we assume the size of G is greater than 5. Consider a 3-path $P_4(s,t,u,v)$ in G. If P_4 is closed, by T133 a triangle (s,t,u) exactly corresponds to a triangle (S,T,U) in G_d. Then suppose P_4 is open. Four points $S(r,s,t)$, $T(s,t,u)$, $U(t,u,v)$, $V(u,v,w)$ are in G_d, where r,s,t,u,v,w are vertices of G. That is, the 3-path P_4 uniquely corresponds to an 3-path (S,T,U,V) in G_d and all the points in the path are distinct. As well an elementary path of length > 2 in G can be decomposed into a set of 3-paths and all of points in those 3-paths are distinct in G_d. Therefore an elementary path xy of G corresponds to an elementary path XY of G_d. Especially if xy is closed, XY also is closed. It is immediate from T145 that an elementary path XY of G_d is a path xy of G. Further if XY is closed, xy also is closed. □

T141: *A graph G has an odd cycle iff G has an odd elementary cycle*.

Proof: If G has an odd elementary cycle C_e, G has an odd cycle C_e. Then assume G has an odd cycle C_o. If C_o is elementary, we have an odd elementary cycle C_o. If C_o is simple, by T143 G has an odd elementary cycle. In general case, the path of C_o may include multi-edges, (i.e., multiple occurrence of the same edge). We can get an odd simple cycle C_s on C_o by the following procedure.

(1) Given an odd cycle C_o. Let C_o be C_s.
(2) If C_s is simple (i.e., with no multi-edges) end.
(3) Choice an arbitrary multi-edge e in C_s.
(4) Remove 2 occurrences of e from C_s and get a new graph C_1.
(5) If C_1 is connected, C_1 is still an odd cycle. let C_1 be C_s. goto (2).
(6) If C_1 is separated into K_1 and a cycle C_2, C_2 is still an odd cycle. Let C_2 be C_s. goto (2).

By definition, C_o have at least three vertices. As the length of C_o is finite, this procedure definitely halts and we obtain an odd simple cycle C_s. Then we can find an odd elementary cycle C_e as above. □

T142: *If a 2-connected graph G has a complete and coherent extension-digraph, then G is transitive* .

Proof: Let D_1 be the complete and coherent extension-digraph of G. By definition, we have an extension-graph G_1 of G and an independent set partition Π for the Π-reduction of G_1 to G. D_1 is a complete plotnikov digraph of G_1 and the edges of D_1 are coherent with Π. G_1 is of course transitive for its plotnikov digraph D_1 is complete. We can construct a complete meta-plotnikov digraph D_0 of G_1, with which the extension-digraph D_1 is coherent by the following procedure.

(1)Given a graph $G(V,E)$, a transitive extension-graph $G_1(V_1,E_1)$ of G, the independent set partition Π of G_1. A complete plotnikov digraph D_1 of G_1 coherent with Π. $|V| = |\Pi|$.
(2)Make a directed multigraph $D_m(V_m,E_m)$ by contracting all vertices of D_1 in an element of Π to a vertex of D_m. $|V_m| = |\Pi|$, $|E_m| = |E_1|$.
(3)Get a directed graph $D_0(V_0,E_0)$ by merging every multi-edges of a vertex pair of D_m into an edge of D_0. $|V_0| = |\Pi| = |V|$.
(4)D_0 is a complete plotnikov digraph of G.

First we contract the vertices of the digraph D_1 and get a multigraph D_m. All of vertices in each independent set of Π are reduced to a vertex in D_m. This makes a multigraph having the same number of edges as G_1, i.e., $|E_m| = |E_1|$, $|V_m| = |\Pi| = |V|$. Since D_1 is complete by hypothesis, D_1 is acyclic and then D_m is also acyclic. Further as we do not eliminate any edges in D_1, the transitivity of D_1 is preserved by the multigraph D_m. Next we reduce all edges in a vertex pair of the multigraph D_m into an edge and get a directed graph D_0. By hypothesis edges of D_1 are coherent with the independent set partition Π, and then all multi-edges in a vertex pair of D_m have the same orientation and are coherent with the corresponding edge in D_0. Obviously D_0 is acyclic and the transitivity of D_m is preserved by D_0, then D_0 is transitively complete. As G_1 is an extension-graph of G_0 and D_1 is the plotnikov digraph of G_1, D_0 is a complete meta-plotnikov digraph of G_1. Hence D_0 is a complete plotnikov digraph of G and then G is transitive. □

T143: *A graph G has an odd simple cycle iff G has an odd elementary cycle* .

Proof: If G has an odd elementary cycle C_e, G has an odd simple cycle C_e. Then assume G has an odd simple cycle C_s. As C_s is an eulerian graph, by T137 we get a distinct elementary cycles set Ce of C_s. Since C_s is odd, the number of edges is odd. Consequently it is obvious that there is at least one odd elementary cycle $C_e \in Ce$. Hence when G has an odd simple cycle, G has an odd elementary cycle. □

T144: *The following three statements are equivalent for a graph G* .
(1)*G has an odd elementary cycle*.
(2)*G has an odd simple cycle*.
(3)*G has an odd cycle*.

Proof: Immediate from T143, T141. □

T145: *Given a 2-connected graph G, the delta graph G_d of G, the pointset partition Π of G_d. A vertex of G one to one corresponds to a pointset in Π and one to many corresponds to points in G_d. An edge in G one to many corresponds to lines in G_d.*

Proof: As G is 2-connected, every vertex y of G is incident with at least two edges xy and yz. Hence G_d has at least one point $p(x,y,z)$ and every vertex y of G has a corresponding pointset $Y \in \Pi$, $p \in Y$. Accordingly a vertex y in G one to one corresponds to a pointset $Y \in \Pi$ and one to many corresponds to points in G_d. By definition, when an edge xy is in G, there are lines p^*q^* in G_d such as $p^*(u^*,x,y) \in X$, $q^*(x,y,v^*) \in Y$, $X,Y \in \Pi$. Hence an edge in G one to many corresponds to lines in G_d. □

T146: *Given a graph G, the delta graph G_d of G, the pointset partition Π of G_d. Suppose points $p_0,p_1 \in P$, $q_0,q_1 \in Q$, $P,Q \in \Pi$. Whenever there exist lines p_0q_0, p_1q_1 in G_d, the lines p_0q_1, p_1q_0 exist in G_d.*

Proof: Without loss of generality we assume that the size of G is greater than 5. As there are lines p_0q_0, p_1q_1, by definition p_0 and q_0 have a common edge x_0y_0 in G, as well p_1 and q_1 have a common edge x_1y_1 in G. Then it comes to be $p_0(w_0,x_0,y_0)$, $q_0(x_0,y_0,z_0)$, $p_1(w_1,x_1,y_1)$, $q_1(x_1,y_1,z_1)$, where $w_0,w_1,x_0,x_1,y_0,y_1,z_0,z_1$ are vertices in G. Moreover $\{p_0, p_1\}$, $\{q_0, q_1\}$ are in the same pointset respectively, i.e., $x_0 = x_1$ and $y_0 = y_1$. Hence it turns to $p_1 = (w_1,x_0,y_0)$, $q_1 = (x_0,y_0,z_1)$. Accordingly it is apparent that there are lines $p_0q_1 = (\mathrm{w}_0,x_0,y_0,z_1)$ and $p_1q_0 = (w_1,x_0,y_0,z_0)$. □

T147: *Given a graph G, the delta graph G_d of G, the pointset partition Π of G_d, the pivot-map M_p(core-map M_c) of G_d. A connected component C_p of $M_p(M_c)$ corresponds to a set S of the pointset pairs of Π and no other components than C_p have pivot-lines(core-lines) belonging to a pointset pair $\in S$.*

Proof: Let Cp_0,Cp_1 be two distinct components of $M_p(M_c)$. Assume there are lines p_0q_0 in Cp_0 and p_1q_1 in Cp_1, where $p_0,p_1 \in P$, $q_0,q_1 \in Q$, $P,Q \in \Pi$. Then by T146, we have lines p_0q_1, p_1q_0 in G_d. This contradicts the hypothesis that Cp_0, Cp_1 are disconnected. Hence all of pivot-lines(core-lines) in a pointset pair (P,Q) is dominated by some particular component of $M_p(M_c)$. That is, a connected component C_p of $M_p(M_c)$ corresponds to an unique pointset-pair set $S = \{(P,Q)\}$ and other components than C_p have no pivot-lines(core-lines) in a pointset pair $(P, Q) \in S$. □

T148: *A subgraph of a graph G induced by the vertices of a core-path in G is a twisted path in G .*

Proof: By definition every delta d_i on a core-path $P_0(p_0,p_1,...,p_n)$ of G is not a pivot, where p_i is a vertex of G and the delta d_i is a 2-path (p_{i-1},p_i,p_{i+1}) in G. Since d_i is not a pivot, the vertex pair (p_{i-1},p_{i+1}) must be joined in G. Hence the subgraph T induced by the vertices of the core-path P_0 has a spanning path $P_0(p_0,p_1,...,p_n)$ and paths $P_1(p_0,p_2,...)$, $\mathrm{P}_2(p_1,p_3,...)$, then T is a twisted path in G. □

T149: *A core-line of the delta graph G_d of size > 2 of a graph G has corresponding triangles in G .*

Proof: A core-line is an edge in a delta graph joining core-points (i.e., non-pivot-points). Suppose a core-line pq in G_d, where $p(s,t,u)$, $q(t,u,v)$ are core-points of G_d and s,t,u,v are vertices in G. By definition we have edges st, tu, uv in G and the core-line pq corresponds to the edge tu. Since the points p, q are non-pivot-points, the vertex pairs (s,u), (t,v) must be joined in G respectively. That is, there are edges su and tv in G. Hence we have triangles (s,t,u) and (t,u,v) in G. The edge tu is included in both triangles, hence the core-line pq has corresponding triangles in G. If the vertex s coincides with v, we have just one triangle (s,t,u). □

T150: *A vertex pair (p,q) in a strongly twisted path T_p is always joined.*

Proof: By definition, a twisted path T_p has a spanning path $P_0(p_0,p_1,...,p_n)$ and paths $P_1(p_0,p_2,...)$, $P_2(p_1,p_3,...)$ such as the alternate vertex sequences of P_0 and $V(P_0) = V(P_1) \cup V(P_2)$. By hypothesis T_p is strongly twisted, hence when there are edges p_0p_{k-1} and $p_{k-1}p_k$, there is an edge p_0p_k. Consequently when a strongly twisted path (p_0,p_k) exist, there exist edges (p_0,p_1), (p_0,p_2), ..., (p_0,p_k). As T_p is connected, for all vertex pair (p,q) has a path (p,q) and then an edge pq always exists. □

T152: *A path in the core-map of the delta graph of a graph G corresponds to a twisted path in G .*

Proof: To prove this, it is enough to show that every 2-path in the core-map M_c of the delta graph G_d of G corresponds to a twisted path in G. T149 says that a core-line of the delta graph G_d has corresponding triangles in G. When there is a 2-path (p,q,r) in M_c, where $p(s,t,u)$, $q(t,u,v)$, $r(u,v,w)$ are core-points and s,t,u,v,w are vertices in G, we have edges st, tu, uv, vw. Since points p,q,r are not pivot-points, vertex pairs (s,u), (t,v), (u,w) must be joined in G respectively. That is, there are edges su, tv, uw in G, then we have triangles (s,t,u), (t,u,v), (u,v,w) in G. Hence whenever there is a 2-core-path (p,q,r), there is a corresponding twisted path (s,t,u,v,w) in G. Especially when t coincides with v, there are two triangles (s,t,u), (u,t,w) and two twisted paths (s,t,u,w) and (s,u,t,w). □

Note: Edges su, tv, uw are not necessarily core-lines. Therefore a connected component of the core-map M_c of G_d is not necessarily a clique of G. The situation is the same for T148.

T160: *A graph G has a pivot-cycle iff G has an OZ-cycle*.

Proof: By T21, if G has an OZ-cycle, G has a pivot-cycle. Then we will show the converse. Let $C_0(p_0,p_1,...,p_{2m})$ be the front-cycle of a pivot-cycle Pv in G. By definition every delta (p_{i-1},p_i,p_{i+1}) on C_0 is a pivot and there is an anti-path aP_i connecting p_{i-1} with p_{i+1} in $N(p_i)$. Consider the case such that for every deltas (p_{i-1},p_i,p_{i+1}) in C_0, $0 \le i \le 2m$,+- is mod $2m+1$: there is not the edge $p_{i-1}p_{i+1}$. Then C_0 is an OZ-cycle. In the other case such that there is an edge $p_{k-1}p_{k+1}$ for some k, $0 \le k \le 2m$, we show that there is always an even zigzag path Z_k connecting p_{k-1} to p_{k+1}. Let aP_k be the anti-path connecting p_{k-1} to p_{k+1}, $aP_k = (p_{k-1},q_1,...,q_w, p_{k+1})$. Since all of vertices in aP_k is in $N(p_k)$, We have an even zigzag path $Z_k = (p_{k-1},p_k,q_1,p_k,..., p_k,q_w,p_k,p_{k+1})$. And we exchange the 2-path (p_{k-1},p_k,p_{k+1}) on C_0 by the even zigzag path Z_k and get an extended odd cycle C_1. We can repeat this operation until there is no short chord (i.e., an edge connecting the end vertices of a 2-path on the cycle) on C_1 and the obtained odd cycle C_1 is the front-cycle of an OZ-cycle in G. □

Rigorous Proof of T160: By T21 if G has an OZ-cycle, G has a pivot-cycle.We show, whenever there is a pivot-cycle $Pv(C_0,C_1)$ in G, there is an OZ-cycle $Z(Z_0,Z_1)$ in G. Let $C_0(p_0,p_1,...,p_{2m})$ be the front-cycle of pivot-cycle Pv. By definition every delta (p_{i-1},p_i,p_{i+1}) on C_0 is a pivot and there is an anti-path aP_i connecting p_{i-1} to p_{i+1} in $N(p_i)$. If all the anti-paths aP_i in the rear-cycle C_1 of Pv are edges $p_{i-1}p_{i+1}$ in $\sim G$, we have $Z_0 = (p_0,p_1,...,p_{2m})$ and $Z_1 = (p_0,p_2,p_4,..., p_{2m-3},p_{2m-1})$. Otherwise, there is a path $P_i = (p_{i,1},p_i,q_{i,1},p_i, q_{i,2},...,p_i,q_{i,k(i)}, p_i,p_{i+1})$ and an anti-path $aP_i = (p_{i-1}, q_{i,1},q_{i,2},...,q_{i,k(i)},p_{i+1})$ for each i, $0 \le i \le 2m$, where $\{q_{i,1}, q_{i,2},...,q_{i,k(i)}\}$ is a connected component in $\sim N(p_i)$. In this case we use $k(i)$ loops $p_i\, p_i$ for the rear cycle Z_1.

$Z_0 = (p_0,$	1
$p_1,q_{1,1},p_1,q_{1,2},p_1,...,q_{1,k(1)},p_1,$	$2k(1)+1$
$p_2,q_{2,1},p_2,q_{2,2},p_2,...,q_{2,k(2)},p_2,$	$2k(2)+1$
...........,	
$p_{2m},q_{2m,1},p_{2m},q_{2m,2},...,q_{2m,k(2m)},p_{2m},$	$2k(2m)+1$
$p_0,q_{0,1},p_0,q_{0,2},p_0,...,q_{0,k(0)}).$	$2k(0)$
$Z_1 = (p_0,$	1
$q_{1,1},q_{1,2},...,q_{1,k(1)},$	$k(1)$
$p_2,p_2,...,p_2,$	$k(2)+1$

$q_{3,1}, q_{3,2}, \ldots, q_{3,k(3)}$, $k(3)$
...,
$q_{2m-1,1}, q_{2m-1,2}, \ldots, q_{2m-1,k(2m-1)}$, $k(2m-1)$
$p_{2m}, p_{2m}, \ldots, p_{2m}$, $k(2m)+1$
$q_{0,1}, q_{0,2}, \ldots, q_{0,k(0)}$, $k(0)$
$p_1, p_1, \ldots, p_1$, $k(1)+1$
$q_{2,1}, q_{2,2}, \ldots, q_{2,k(2)}$, $k(2)$
....,
$p_{2m-1}, p_{2m-1}, \ldots, p_{2m-1}$, $k(2m-1)+1$
$q_{2m,1}, q_{2m,2}, \ldots, q_{2m,k(2m)}$, $k(2m)$
$p_0, p_0, \ldots, p_0$). $k(0)$

the length of $Z_0, Z_1 = 2\sum_{i=0}^{2m} k(i) + 2m + 1$ = odd. I.e., Z_0 and Z_1 are odd cycles and their vertex sets coincide with $V(C_1)$ of the pivot-cycle $Pv(C_0,C_1)$. Hence whenever G has a pivot-cycle $Pv(C_0,C_1)$, G has an OZ-cycle $Z(Z_0,Z_1)$, $|Z| = |Z_0| = |Z_1| = |C_1|$. □

T161: *A graph G has a pivot-cycle iff G has an elementary pivot-cycle*.

Proof: See Gallai, 1967 [3]. □

T170✯ *For every berge graph G_0, there exists a perfect graph G_1 obtained by adding edges to G_0 satisfying the following inequality*: *maximum clique size of $G_1 \leq$ maximum clique size of G_0*.✯

Proof: Pending...

T171: *Every vertex induced subgraph of a berge graph is a berge graph*.

Proof: Let *Bs* be an induced subgraph of a berge graph *B*. Whenever *Bs* has an odd hole, *B* has an odd hole. As well whenever *Bs* has an odd anti-hole, *B* has an odd anti-hole. This contradicts the definition of berge graph. Hence a subgraph of a berge graph has neither odd holes nor odd anti-holes. I.e., a subgraph of a berge graph is a berge graph. □

T172: *A perfect graph is a berge graph*.

Proof: By T3, a perfect graph *G* and the complement has no odd holes. Assume *G* has an odd anti-hole *aH*. Consider the complement *aG* of *G*. By T1, *aG* is perfect. As *G* has an odd anti-hole *aH*, *aG* has an odd hole *H*. I.e., a perfect graph *aG* has an odd hole *H*. This contradicts T3, hence a perfect graph *G* has neither odd holes nor odd anti-holes. As well a perfect graph *aG* has neither odd holes nor odd anti-holes I.e., a perfect graph is a berge graph. □

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Appendix

Gallai’s Gamma Table

Minimally Intransitive Graph Collection