THE NP-COMPLETENESS OF THE HAMILTONIAN CYCLE PROBLEM IN PLANAR DIGRAPHS WITH DEGREE BOUND TWO

J. PLESŃIK

Komensky University, Department of Mathematics, 816 31 Bratislava, Czechoslovakia

Received 21 August 1978

Combinatorial problems, computational complexity

1. Introduction

The notation and terminology follow Harary [4]. A hamiltonian cycle in a graph or digraph is a cycle containing all the points. Thus any such cycle has p points as well as p lines (arcs) if the graph (digraph) has p points. No elegant characterization of the graphs or digraphs which possess hamiltonian cycles exists, although the problem is at least one hundred years old [4]. As the problem is a special case of the famous traveling salesman problem, it is interesting also from the computational viewpoint. However, in [5], the problem can be found as NP-complete, which means that no polynomial time (in the size of input) algorithm is known and probably none can exist. Garey, Johnson, and Stockmeyer [2] have proved that the problem remains NP-complete also in the case of undirected graphs with degrees at most 3. Then Garey, Johnson, and Tarjan [3] have shown the NP-completeness under constraints that the undirected graphs are planar, cubic, and 3-connected. This is obviously the best possible result as for degree bound. Replacing every line uv of undirected graphs with two arcs uv and vu, they have simultaneously settled the NP-completeness in the case of planar digraphs with indegrees and outdegrees at most 3. Here we show that the last result can be still improved as follows: The hamiltonian cycle problem is NP-complete even in the case of planar digraphs with indegrees and outdegrees at most 2. Clearly, the bound two on degrees is the best possible.

2. Proof of the result

Our proof is analogous to that of Garey, Johnson, and Tarjan for the undirected case [3]. In some details it is even less complicated. As the problem is obviously in the class NP, it remains to prove the completeness. For this purpose we shall polynomially

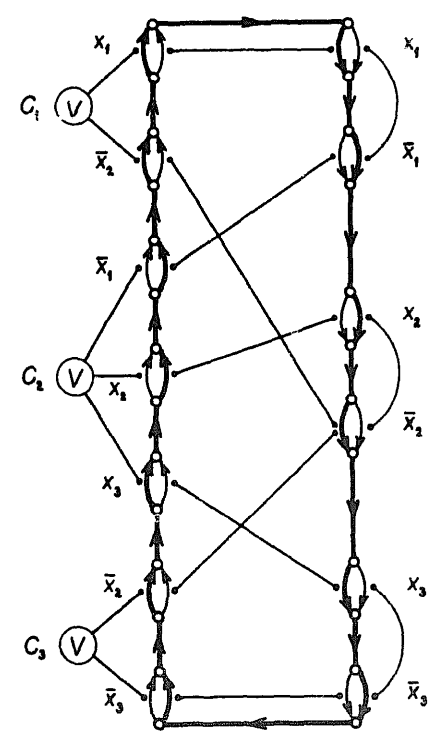

Fig. 1. Construction of G for $F = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$, $F(1, 0, 1) = 1$.

reduce the problem of satisfiability of a boolean formula F in conjunctive normal form to our problem. We suppose that $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where clause $C_j = p_{j1} \vee p_{j2} \vee \cdots \vee p_{js_j}$ ($s_j \geqslant 2$) with literals $p_{jk} \in \{x_1, x_2, ..., x_n, \bar{x}_1, \bar{x}_2, ..., \bar{x}_n\}$. Here x_i denotes an atomic variable and $\bar{x}_i$ its negation. For the form F we shall construct a planar digraph G with indegrees 1 or 2 and outdegrees 2 or 1, respectively, and with the property that F is satisfiable if and only if G has a hamiltonian cycle. Our construction will be illustrated with aid of Fig. 1 where we have the basic scheme of G for $F = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$. (Here $m = n = 3$, $s_1 = 2$, $s_2 = 3$ and $s_3 = 2$.)

At first we assign to F a directed cycle Z of length $2(s_1 + s_2 + \cdots + s_m + 2n)$ with alternately placed simple arcs and double arcs. It is assumed that

(a) to every clause C_j a segment of Z is assigned which contains s_j double arcs ($2s_j$ points),

(b) a segment containing two double arcs (4 points) is assigned to each pair of variables x_i, $\bar{x}_i$, and

(c) all double arcs corresponding to literals of clauses are in one (left) section of Z and those corresponding to variables are also in one (right) section of Z.

The directed multigraph Z is a skeleton which we shall extend to a digraph G with aid of special couplings (cf. Fig. 1).

We shall require that the value of x_i (or $\bar{x}_i$) is true if a given hamiltonian cycle of Z contains the left arc (in a diagram like to that in Fig. 1) from every pair of arcs assigned to x_i (or $\bar{x}_i$, respectively). For example, the hamiltonian cycle depicted in Fig. 1 with a thick line gives $x_1 = 1$ (true), $x_2 = 0$ (false), and $x_3 = 1$ (true). Suppose for a moment the existence of a genie securing that every hamiltonian cycle of Z:

(i) contains at least one left arc in every C_j,

(ii) passes through either all left arcs assigned to x_i (or $\bar{x}_i$) or all right arcs assigned to x_i ($\bar{x}_i$, respectively), and

(iii) never passes simultaneously through a left (right) arc of x_i and a left (right, respectively) arc of $\bar{x}_i$.

We see that Z with such a genie has a hamiltonian cycle if and only if F is satisfiable. In what follows the genie is replaced by certain couplings (see Fig. 1) and these will be replaced by proper digraphs.

The first coupling called 'k-input or' connects k

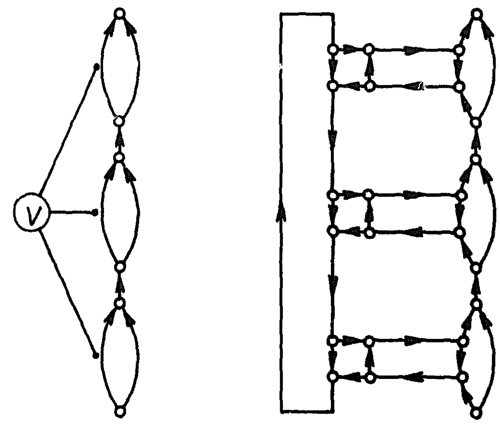

Fig. 2. '3-input or' and its digraph realization.

($k \geqslant 2$) arcs and ensures that at least one of them must occur in any hamiltonian cycle of Z, what means that each C_j will be satisfied (the condition (i)). To realize this idea, we need construct a digraph which will work as 'k-input or'. In Fig. 2 we have a realization of '3-input or'. The reader will easily construct 'k-input or' for any $k \geqslant 2$ (supposing that F has exactly three literals per clause, he has nothing to do). In our realization each coupled arc is replaced by three arcs e_1, e_2, e_3 in such a way that the original arc occurs in a hamiltonian cycle of Z if and only if e_1 and e_3 simultaneously occur in a hamiltonian cycle of the extension of Z. This can be seen with aid of Fig. 3.

The second coupling called 'exclusive-or line' ensures that exactly one of the two coupled arcs must occur in any hamiltonian cycle of Z. Using a number of these couplings (see Fig. 1), we fulfill the conditions (ii) and (iii). A digraph realization of 'exclusive-or line' gives Fig. 4. One sees that a coupled arc e occurs in a hamiltonian cycle if and only if the arcs e_1, e_3, and e_5 simultaneously do so. On the other hand, e does not occur in a hamiltonian cycle if and only if e_2 and e_4 occur.

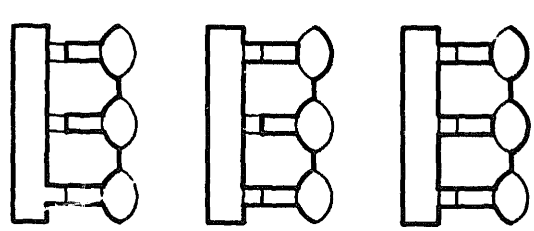

Fig. 3. Possible local states for '3-input or' (symmetric cases not shown).

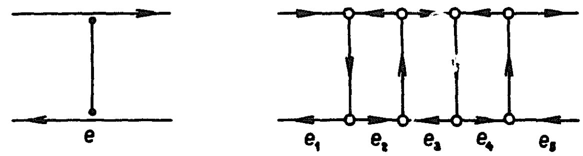

Fig. 4. 'Exclusive-or line' and its digraph realization.

To ensure planarity of the constructed digraph, we must solve the crossing of 'exclusive-or lines'. However, the reader can easily verify that this can be done as in Fig. 5.

The construction is completed. Our digraph G is obviously planar and every its point has either indegree 1 and outdegree 2 or indegree 2 and outdegree 1. We have seen that the genie can be fully replaced. Hence the form F is satisfiable if and only if G has a hamiltonian cycle. It can be easily verified that G was constructed in polynomial (of $m + n$) time. This completes the proof.

3. Remarks

Note that our graph G contains many arcs uv where u has indegree 2 and v has outdegree 2 (so called permanent arcs as they occur in any hamiltonian cycle of G). Every permanent arc can be shrinked to single point with indegree 2 and outdegree 2. This process is also reversible.

Choose two permanent arcs from the same face (e.g. the upper and the lower arcs in Fig. 1) and connect them by the 'exclusive-or line'. The obtained digraph will have a hamiltonian path if and only if G will have a hamiltonian cycle. Hence the problem of directed hamiltonian path remains NP-complete for our class of digraphs (the planar digraphs with degrees 1-In and 2-Out or 2-In and 1-Out).

We shall mention still two problems which remain NP-complete for this class of digraphs.

A point-disjoint path cover of a digraph is a collection of point-disjoint paths which covers all the points. The problem of deciding whether there is such a collection of k paths is NP-complete [1].

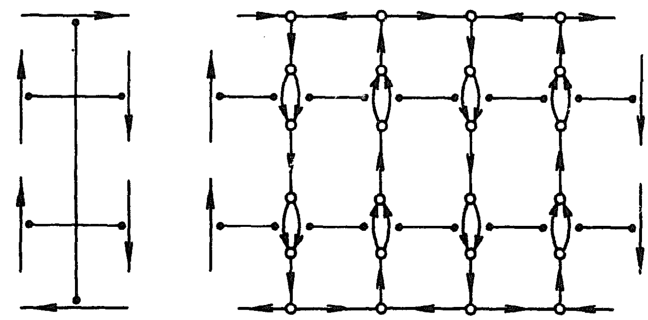

Fig. 5. Crossing of exclusive-or lines and a planar realization.

Namely, for $k = 1$ it is precisely the hamiltonian path problem.

Finally, consider the following reachability problem. Find a spanning subgraph G_1 of a given digraph G such that G_1 has the minimum possible number of arcs and the same reachability as G (i.e., G_1 has a directed $u-v$ path whenever G has). Sometimes G_1 is called a transitive irreducible kernel of G [6] or minimal equivalent subgraph of G [7]. This problem contains the hamiltonian cycle problem as a special case. Another version of the problem is to find a minimum strongly connected spanning subgraph G_1 of a digraph G.

References

[1] F.T. Boesch and J.F. Gimpel, Covering the points of a digraph with point-disjoint paths and its application to code optimization, J. ACM 24 (1977) 192–198.

[2] M.R. Garey, D.S. Johnson and L.J. Stockmeyer, Some simplified NP-complete graph problems. Theoret. Comput. Sci. 1 (1976) 237–267.

[3] M.R. Garey, D.S. Johnson and R.E. Tarjan, The planar hamiltonian circuit problem is NP-complete, SIAM J. Comput. 5 (1976) 704–714.

[4] F. Harary, Graph theory (Addison–Wesley, Reading, MA, 1969).

[5] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller and J.W. Thatcher, eds., Complexity of computer computations (Plenum Press, New York, 1972) 85–104.

[6] H. Noltemeier, Reduktion von Präzedenzstrukturen, Z. Operations Res. 20 (1976) 151–159.

[7] S. Sahni, Computationally related problems, SIAM J. Comput. 3 (1974) 262–279.